

PARABOLIC DYNAMICS AND ANISOTROPIC BANACH SPACES

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ABSTRACT. We investigate the relation between the distributions appearing in the study of ergodic averages of parabolic flows (e.g. in the work of Flaminio-Forni) and the ones appearing in the study of the statistical properties of hyperbolic dynamical systems (i.e. the eigendistributions of the transfer operator). In order to avoid, as much as possible, technical issues that would cloud the basic idea, we limit ourselves to a simple flow on the torus. Our main result is that, roughly, the growth of ergodic averages of a parabolic flows is controlled by the eigenvalues of a suitable transfer operator associated to the renormalising dynamics. The conceptual connection that we illustrate is expected to hold in considerable generality.

1. INTRODUCTION

In the last decade, *distributions* have become increasingly relevant both in parabolic and hyperbolic dynamics. On the *parabolic dynamics* side consider, for example, the work of Forni and Flaminio-Forni [20, 21, 16, 17, 18] on ergodic averages and cohomological equations for horocycle flows; on the *hyperbolic dynamics* side it suffices to mention the study of the transfer operator through anisotropic spaces, started with [38].¹

Since a typical approach to the study of *parabolic dynamics* is the use of renormalization techniques,² where the renormalizing dynamics is typically a *hyperbolic dynamics*, several people have been wondering on a possible relation between such two classes of distributions. Early examples of such line of thought can be found in Cosentino [11, Section 3] and Otal [39].

In this paper we show that the distributional obstructions discovered by Forni and the distributional eigenvectors of certain transfer operators are tightly related, to the point that, informally, we could say that they are exactly the same.

In order to present our argument in the simplest possible manner, instead of trying to develop it for the horocycle flow versus the geodesic flow (which would

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¹ But see, e.g., [40, 41, 35, 34] for earlier related results.

² Typical examples are circle rotations [43, 44], interval exchange maps via Teichmüller theory [45, 23], horocycle flow [21, 16].

require a much more technical framework), we consider a very simple example that, while preserving the main ingredients of the horocycle-geodesic flow setting, allows to easily illustrate the argument. Yet, our example is not rigid (morally it corresponds to looking at manifolds of non constant negative curvature). So, notwithstanding its simplicity, it shows the flexibility of our approach, which has the potential of being greatly generalised. On the other hand, it covers only the case of periodic renormalization. Indeed, if the renormalizing dynamics are non linear, then it is not very clear how to define a good moduli space on which to act. The extension of our approach to the non periodic case remains an open problem.

Let us describe a bit more precisely our setting (see Section 2 for the exact, less discursive, description). As parabolic dynamics, we consider a flow ϕ_t , over $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, generated by a vector field $V \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2, \mathbb{R}^2)$, $\alpha \in \mathbb{R}_+$, such that, for all $x \in \mathbb{T}^2$, $V(x) \neq 0$. As hyperbolic dynamics, we consider a transitive Anosov map $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$, $r > 1 + \alpha$. By definition of Anosov map for all $x \in \mathbb{T}^2$ we have $T_x \mathbb{T}^2 = E^s(x) \oplus E^u(x)$, where we used the usual notation for the stable and unstable invariant distributions.³ Since we want the latter system to act as a renormalizing dynamics for the former, we require,

$$(1.1) \quad \forall x \in \mathbb{T}^2, V(x) \in E^s(x).$$

One might wonder which kind of flows admit the property (1.1) for some Anosov map $F \in \mathcal{C}^r$. Here is a partial answer whose proof can be found in Appendix A.

Lemma 1.1. *If a $\mathcal{C}^{1+\alpha}$, $\alpha > 0$, flow ϕ_t , without fixed points, satisfies (1.1) for some Anosov map $F \in \mathcal{C}^r$, $r \geq 1 + \alpha$, then it is topologically conjugated to a rigid rotation with rotation number ω such that*

$$(1.2) \quad b\omega^2 + (a - d)\omega - c = 0$$

for some $a, b, c, d \in \mathbb{Z}$ such that $ad - cb = 1$.

Each $\mathcal{C}^{1+\alpha}$, $\alpha \geq 1$, flow ϕ_t without fixed points, or periodic orbits, it is topologically conjugated to a rigid rotation. If the rotation number satisfies (1.2) and $\alpha \geq 2$, then ϕ_t satisfies (1.1) for some Anosov map $F \in \mathcal{C}^\beta$, for each $\beta < \alpha$.

Remark 1.2. *Even though the above Lemma shows that it is always possible to reduce our setting to a linear model by a conjugation, such conjugation is typically of rather low regularity. We will see shortly that considering F , and related objects, of high regularity is essential for the questions we are interested in. It is not obvious to us how to characterise the flows for which (1.1) holds, for very smooth F . Yet, such a condition clearly singles out some smaller class of flows (compared to Lemma 1.1) to which our theory applies. Note however that there are plenty of examples, see Appendix B*

Equation (1.1) implies that the trajectories $\{\phi_t(x)\}_{t \in (a,b)}$ are pieces of the stable manifolds for the map F . Thus, we can define implicitly a function $\nu_n \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2, \mathbb{R})$ such that

$$(1.3) \quad D_x F^n V(x) = \nu_n(x) V(F^n(x)),$$

³ Here “distribution” refers to a field of subspaces in the tangent bundle and has nothing to do with the meaning of “distribution” as generalised functions previously used. This is an unfortunate linguistic ambiguity for which we bear no responsibility.

where $|\nu_n| < C_{\#} \lambda^{-n}$ for some $\lambda > 1$. Without loss of generality we assume that F preserves the orientation of the invariant splittings, i.e. $\nu_n > 0$ (if not, use F^2).

Given the hypothesis (1.1) it is natural to ask, at least, that, for each $x \in \mathbb{R}^2$, the flow is regular with respect to the time coordinate i.e.

$$(1.4) \quad \phi_{(\cdot)}(x) \in \mathcal{C}^r.$$

In fact, we will use a slightly stronger hypotheses, see Definition 2.3 and Remark 2.4.

The reader may complain that the parabolic nature of the dynamics ϕ_t it is not very apparent. Indeed, a little argument is required to show that $\|D_x \phi_t\|$ can grow at most polynomially in t , see Section 4.1.

Let us detail an easy consequence of (1.3). If, for each $n \in \mathbb{N}$, we define $\eta_n \in \mathcal{C}^{1+\alpha}(\mathbb{T}^2 \times \mathbb{R}, \mathbb{T}^2)$ by $\eta_n(x, t) = F^n(\phi_t(x))$, we have

$$\begin{cases} \frac{d}{dt} \eta_n(x, t) = D_{\phi_t(x)} F^n V(\phi_t(x)) = \nu_n(\phi_t(x)) V(\eta_n(x, t)) \\ \eta_n(0) = F^n(x). \end{cases}$$

It is then natural to define the time change⁴

$$(1.5) \quad \tau_n(x, t) = \int_0^t ds \nu_n(\phi_s(x)),$$

and introduce the function $\gamma_n \in \mathcal{C}^{1+\alpha}$ by $\gamma_n(\tau_n(x, t), x) = \eta_n(x, t)$. Then

$$(1.6) \quad \begin{cases} \frac{d}{ds} \gamma_n(x, s) = V(\gamma_n(x, s)) \\ \gamma_n(x, 0) = F^n(x). \end{cases}$$

By the uniqueness of the solution of the above ODE, it follows, for all $s \in \mathbb{R}$

$$\phi_s(F^n(x)) = \gamma_n(x, s) = \eta_n(\tau_n^{-1}(x, s), x) = F^n(\phi_{\tau_n^{-1}(x, s)}(x)).$$

In other words, the image under F^n of a piece of trajectory, is the reparametrization of a (much shorter) piece of trajectory:

$$(1.7) \quad F^n(\phi_t(x)) = \phi_{\tau_n(x, t)}(F^n(x)).$$

The above is the basic renormalization equation for the flow ϕ_t that we will use in the following.

Note that, by Lemma (1.1) and Furstenberg [22], the flow is uniquely ergodic, since its Poincaré map is uniquely ergodic. Let μ be the unique invariant measure.

By unique ergodicity, given $g \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$, $\frac{1}{t} \int_0^t ds g \circ \phi_s(x)$ converges uniformly to $\mu(g)$. We have thus naturally arrived at our

First question: How fast is the convergence to the ergodic average?

The question is equivalent to investigating the precise growth of the functionals $H_{x,t} : \mathcal{C}^r \rightarrow \mathbb{R}$ defined by

$$(1.8) \quad H_{x,t}(g) := \int_0^t ds g \circ \phi_s(x).$$

Of course, if $\mu(g) \neq 0$, then $H_{x,t}(g) \sim \mu(g)t$, but if $\mu(g) = 0$, then we expect a slower growth.

⁴ By construction, for each $x \in \mathbb{T}^2$ and $n \in \mathbb{N}$, $\tau_n(x, t)$ is a strictly increasing function of t , and hence globally invertible. We will use the, slightly misleading, notation $\tau_n^{-1}(x, \cdot)$ for the inverse.

Remark 1.3. *Note that the growth rate of an ergodic average it is not a topological invariant, hence the fact that our systems can be topologically conjugated to a linear model, as stated in Lemma 1.1, it is not of much help.*

In the work of Flaminio-Forni [16] is proven that the functional (1.8), there defined for the horocycle flow on a surface of constant negative curvature, has growth controlled by a finite number of obstructions. That is, the growth is slower if the function g belongs to the kernel of certain functionals. The remarkable discovery of Forni is that such obstructions cannot be expected, in general, to be measures: they are *distributions*.⁵

In analogy with the above situation, one expects that also in our simple model there exist a finite number of functionals $\{O_i\}_{i=1,\dots,N_1} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})'$, and a corresponding set $\{\alpha_i\}_{i=1,\dots,N_1}$ of decreasing numbers $\alpha_i \in [0, 1]$ such that if $O_j(g) = 0$ for all $j < i$ and $O_i(g) \neq 0$, then $H_{x,t}(g) = \mathcal{O}(t^{\alpha_i})$. As we mentioned just after equation (1.8), $O_1(g) = \mu(g)$ with $\alpha_1 = 1$.

Next, suppose that $O_i(g) = 0$ for all $i \leq N_1$ and $\alpha_{N_1} = 0$. That is, $H_{x,t}(g)$ remains bounded. It is then convenient to introduce, for each $T \in \mathbb{R}_>$, the new functionals $\overline{H}_T : \mathcal{C}^r \rightarrow \mathcal{C}^{1+\alpha}$ defined by, for all $x \in \mathbb{T}^2$,

$$(1.9) \quad \overline{H}_T(g)(x) := - \int_0^T dt \chi \circ \tau_{n_T}(x, t) g \circ \phi_t(x),$$

where $n_T + 1 = \inf\{n \in \mathbb{N} : \inf_x \tau_n(x, T) \leq 1\}$ and $\chi \in \mathcal{C}^r(\mathbb{R}_>, [0, 1])$ is a fixed function such that $\chi(0) = 1$, $\frac{d^k}{dt^k} \chi(0) = 0$ for all $0 \leq k \leq r$, and $\chi(s) = 0$ for all $s \geq 1$. Such a function χ can be thought as a “smoothing” of $\chi_* \circ \tau_{n_T} = \max\{0, \frac{T-s}{T}\}$. Unfortunately, we cannot use χ_* because such a choice would create catastrophic problems later on (e.g., in the decomposition discussed in Lemma 4.5), yet the reader can substitute χ_* to χ to have an intuitive idea of what is going on. In particular, for χ_* the following assertion, proved in Section 4.2, would be rather trivial.

Lemma 1.4. *For each $g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$, $r \geq 1 + \alpha$, such that $O_i(g) = 0$ for all $i \in \{1, \dots, N_1\}$, there exists $C > 0$ such that*

$$\sup_T \|\overline{H}_T(g)\|_{L^\infty}^\infty \leq C$$

$$\lim_{T \rightarrow \infty} \left\| \overline{H}_T(g) \circ \phi_t(x) - \overline{H}_T(g)(x) - \int_0^t g \circ \phi_s(x) ds \right\|_{L^\infty} = 0.$$

Then, the sequence $\{\overline{H}_T(g)\}_{T \in \mathbb{R}_>}$ is weakly compact in $(L^1)' = L^\infty$. Let $h \in L^\infty$ be an accumulation point of $\{\overline{H}_T(g)\}_{T \in \mathbb{R}_>}$, then, for all $\varphi \in L^1(\mathbb{T}^2 \times \mathbb{R})$ with compact support,

$$(1.10) \quad \int_{\mathbb{T}^2 \times \mathbb{R}} dx dt \varphi(x, t) \left(h \circ \phi_t(x) - h(x) - \int_0^t g \circ \phi_s(x) ds \right) = 0.$$

Thus $h \circ \phi_t - h = \int_0^t g \circ \phi_s ds$ for almost all x and t . It then follows, eventually by changing h on a zero measure set, that h is almost surely derivable in the flow direction. Hence,

$$(1.11) \quad g(x) = \langle V(x), \nabla h(x) \rangle.$$

⁵ A part, of course, for the first that, as above, is the invariant measure μ .

That is, g is a measurable coboundary.

The above fact is of a debatable interest; on the contrary, the existence of more regular solutions of (1.11) is of considerable interest and it plays a role in establishing many relevant properties (see [32, Sections 2.9, 19.2]). Hence our

Second question: How regular are the solutions of the cohomological equation (1.11)?

Following Forni again, we expect that there exist finitely many distributional obstructions $\{O_i\}_{i=N_1+1}^{N_2} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})'$ and a set of increasing numbers $\{r_i\}_{i=N_1+1}^{N_2}, r_i \in (0, 1 + \alpha)$ such that, if $O_j(g) = 0$ for all $j < i$ and $O_i(g) \neq 0$, then $h \in \mathcal{C}^{r_i}$.

Remark 1.5. *Note that in the present context, as the flow is only $\mathcal{C}^{1+\alpha}$, it is not clear if it makes any sense to look at coboundaries better than $\mathcal{C}^{1+\alpha}$. This reflects the fact that if one looks at the horocycle flows on manifolds of non constant negative curvature, then the associate vector field is, in general, not very regular. On the other hand rigidity makes not so interesting our simple example when both foliations are better than \mathcal{C}^2 , [24, Corollary 3.3]. We will therefore limit ourself to finding distributions that are obstruction to Lipschitz coboundaries, i.e. if $O_i(g) = 0$ for all $i \leq N_2$, then h is Lipschitz. We believe this to be more than enough to illustrate the scope of the method.*

The goal of this paper is to prove the above facts by studying transfer operators associated to F , acting on appropriate spaces of distributions. In fact, we will show that the above mentioned obstructions $\{O_i\}$ can be derived from the eigenvectors of an appropriate transfer operator associated to F . As announced, this discloses the connection between the appearance of distributions in two seemingly different fields of dynamical systems.

Remark 1.6. *As already mentioned, in our model ϕ_t plays the role of the horocycle flow, while F the one of the geodesic flow. It is important to notice that most of the results obtained for the horocycle flows (and Flaminio-Forni's results in particular) rely on representation theory, thus requiring constant curvature of the space. In our context, this would correspond to the assumption that F is a toral automorphism and ϕ_t a rigid translation. One could then do all the needed computations via Fourier series (if needed, see Appendix B.1 for details). It is then clear that extending our approach to more general parabolic flows, e.g. horocycle flows, (which should be quite possible using the results on flows by [26, 15, 14]) would allow to treat cases of variable negative curvature, and, more generally, cases where the tools of representation theory are not available or effective, whereby greatly extending the scope of the theory.*⁶

The plan of the paper is as follow: in section 2 we state our exact assumptions, outline our reasoning and state precisely our results, assuming lemmata and constructions which are explained later on. Section 3 is devoted to our first question and proves our Theorem 2.8 concerning the distributions arising from the study of the ergodic averages. Section 4 deals with our second question and proves our Theorem 2.12 dealing with the distributions arising from the study of the regularity of the cohomological equation. In the Appendices A we provide the details for some

⁶ See however [8] for an alternative approach that avoids representation theory, although at the price of being able to probe the obstructions only up to a given level of regularity, independently of r .

facts mentioned in the introduction without proof. In appendix [B](#) we work out explicitly the simplest example and comment on some class of examples to which our theory applies. In the Appendices [C](#) and [D](#) we recall the definition of the various functional spaces needed in the following (adapted to the present simple case).

Notation. When convenient, we will use $C_{\#}$ to designate a generic constant, depending only on F and ϕ_1 , whose actual value can change from one occurrence to the next.

2. DEFINITIONS AND MAIN RESULTS

In this section we will introduce rigorously the model loosely described in the introduction and explicitly state our results. Unfortunately, this requires quite a bit of not so intuitive notations and constructions, which call for some explanation. The experienced reader can jump immediately to Theorems [2.8](#), [2.12](#) but we do not recommend it in general.

Let $\alpha, r \in \mathbb{R}_{>}$ with $r > 1 + \alpha$.

We start by recalling the definition of C^r Anosov map of the torus.

Definition 2.1. Let $F \in C^r(\mathbb{T}^2, \mathbb{T}^2)$ where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The map is called Anosov if there exists two continuous closed nontrivial transversal cone fields $C^{u,s} : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ which are strictly DF -invariant. That is, for each $x \in \mathbb{T}^2$,

$$(2.1) \quad \begin{aligned} D_x F C^u(x) &\subset \text{Int } C^u(F(x)) \cup \{0\} \\ D_x F^{-1} C^s(x) &\subset \text{Int } C^s(F^{-1}(x)) \cup \{0\}. \end{aligned}$$

In addition, there exists $C > 0$ and $\lambda > 1$ such that, for all $n \in \mathbb{N}$,

$$(2.2) \quad \begin{aligned} \|D_x F^{-n} v\| &> C \lambda^n \|v\| && \text{if } v \in C^s(x); \\ \|D_x F^n v\| &> C \lambda^n \|v\| && \text{if } v \in C^u(x). \end{aligned}$$

It is well known that the above implies the following, seemingly stronger but in fact equivalent [\[32\]](#), definition

Definition 2.2. Let $F \in C^r(\mathbb{T}^2, \mathbb{T}^2)$. The map is called Anosov if there exists a DF -invariant $C^{1+\alpha}$, $r - 1 \geq \alpha > 0$, splitting $T_x M = E^s(x) \oplus E^u(x)$ and constants $C, \lambda > 1$ such that for $n \geq 0$

$$(2.3) \quad \begin{aligned} \|DF^n v\| &< C \lambda^{-n} \|v\| && \text{if } v \in E^s; \\ \|DF^n v\| &< C \lambda^n \|v\| && \text{if } v \in E^u. \end{aligned}$$

As already mentioned we assume that the stable distribution E^s is orientable and that F preserves such an orientation. Further note that, since F is topologically conjugated to a toral automorphism [\[32, Theorem 18.6.1\]](#), F is topologically transitive.

Next, we consider a flow ϕ_t generated by a vector field V satisfying the following properties.

Definition 2.3. Let the vector field V be such that

- (i) $V \in C^{1+\alpha}(\mathbb{T}^2, \mathbb{R}^2)$;
- (ii) $\|V\| \in C^r(\mathbb{T}^2, \mathbb{R}_{>})$;
- (iii) for all $x \in \mathbb{T}^2$, $V(x) \neq 0$;
- (iv) for all $x \in \mathbb{T}^2$, $V(x) \in E^s(x)$.

Remark 2.4. Note that Definitions 2.3-(ii) and (iii) imply condition (1.4) since, being $F \in \mathcal{C}^r$, so are the stable leaves [32]. In fact, Definition 2.3-(ii) essentially implies that we are just considering \mathcal{C}^r time reparametrizations of the case $\|V\| = 1$. Hence, we are treating all the \mathcal{C}^r reparametrizations on the same footing. This is rather convenient although not so deep in the present context. Yet, it could be of interest if the present point of view could be extended to the study of the mixing speed of the flow. Indeed, there is a scarcity of results on reparametrization of parabolic flows (see [19] for recent advances).

Remembering (1.7), (1.5) and using the definition (1.8),

$$\begin{aligned} H_{x,t}(g) &= \int_0^t ds g \circ F^{-n} \circ \phi_{\tau_n(x,s)}(F^n(x)) \\ (2.4) \quad &= \int_0^{\tau_n(x,t)} ds_1 \nu_n(F^{-n} \circ \phi_{s_1}(F^n(x)))^{-1} g \circ F^{-n} \circ \phi_{s_1}(F^n(x)). \end{aligned}$$

It is then natural to introduce the *transfer operator* $\mathcal{L}_F \in L(\mathcal{C}^0, \mathcal{C}^0)$,⁷

$$\begin{aligned} \mathcal{L}_F(g) &:= (\nu_1 \circ F^{-1})^{-1} g \circ F^{-1} = g \circ F^{-1} \frac{\|V\|}{\langle \widehat{V}, (DF\widehat{V}) \circ F^{-1} \rangle \|V\| \circ F^{-1}} \\ (2.5) \quad &= g \circ F^{-1} \frac{\|V\|}{\|DFV\| \circ F^{-1}} = g \circ F^{-1} \frac{\|DF^{-1}V\|}{\|V\| \circ F^{-1}}, \end{aligned}$$

where $\widehat{V}(x) = \|V(x)\|^{-1}V(x)$. We can now write

$$(2.6) \quad H_{x,t}(g) = \int_0^{\tau_n(x,t)} ds_1 (\mathcal{L}_F^n g)(\phi_{s_1}(F^n(x))) = H_{F^n(x), \tau_n(x,t)}(\mathcal{L}_F^n g).$$

The above formula is quite suggestive: if we fix $n = n_t(x)$ such that $\tau_{n_t}(x, t)$ is of order one, then, for each $x \in \mathbb{T}^2$ and $t \in \mathbb{R}_+$, $H_{x,t}(g)$ is expressed in terms of very similar functionals of $\mathcal{L}_F^{n_t} g$. In particular, such functionals are uniformly bounded on \mathcal{C}^0 . It is thus natural to expect that, to address the questions put forward in the introduction, it suffices to understand the behavior of \mathcal{L}_F^n for large n . This obviously is determined by the spectral properties of \mathcal{L}_F .

Unfortunately, it is well known that the spectrum of \mathcal{L}_F depends strongly on the Banach space on which it acts. For example, in the trivial case when F is a toral automorphism and ϕ_t a rigid translation with unit speed, $e^{-h_{\text{top}}} \mathcal{L}_F$ acting on L^2 is an isometry,⁸ hence the spectrum of \mathcal{L}_F consists of the circle of radius $e^{h_{\text{top}}}$; while, if we consider \mathcal{L}_F acting on \mathcal{C}^r , the spectral radius will be given by $e^{(r+1)h_{\text{top}}}$.

This seems to render completely hopeless the above line of thought.

Yet, as mentioned in the introduction, it is possible to define norms $\|\cdot\|_{p,q}$ and associated anisotropic Banach spaces $\mathcal{C}^{p+q} \subset \mathcal{B}^{p,q} \subset (\mathcal{C}^q)'$, $p \in \mathbb{N}^*, q \in \mathbb{R}_+$, $p+q \leq r$, such that each transfer operator with \mathcal{C}^r weight can be continuously extended to $\mathcal{B}^{p,q} = \overline{\mathcal{C}^r}^{\|\cdot\|_{p,q}}$. The above are spaces of distributions (a fact that the reader might find annoying) but, under mild hypotheses on the weight used in the operators, several remarkable properties hold true

- i) a transfer operator (with \mathcal{C}^r weight) extends by continuity from \mathcal{C}^r to a bounded operator on $\mathcal{B}^{p,q}$;

⁷ Given a map F , in general a transfer operator associated to F has the form $\varphi \rightarrow \varphi \circ F^{-1} e^\phi$ for some function ϕ . Normally, the factor e^ϕ is called the *weight* while ϕ is the *potential*, [2].

⁸ As usual h_{top} stands for the topological entropy of the map F we are considering.

- ii) such a transfer operator is a quasi-compact operator with a simple maximal eigenvalue;
- iii) the essential spectral radius of the transfer operator decreases exponentially with $\inf\{q, p\}$.⁹

The possibility to make the essential spectrum arbitrarily small, by increasing p and q , will play a fundamental role in our subsequent analysis. Unfortunately, a further problem now arises: the weight of \mathcal{L}_F contains the vector field V which, by hypothesis, is only $\mathcal{C}^{1+\alpha}$. Hence \mathcal{L}_F leaves \mathcal{C}^r invariant only for $r \leq 1 + \alpha$ (exactly the range in which we are not interested). Again it seems that we cannot use our strategy in any profitable manner.

Yet, such a problem has been overcome as well, e.g., in [28]. The basic idea is to extend the dynamics F to the oriented Grassmannian. Indeed, looking at (2.5), it is clear that the weight can be essentially interpreted as the expansion of a volume form. The simplest idea would then be to let the dynamics act on one forms on \mathbb{T}^2 . Unfortunately, the length cannot be written exactly as a volume form on \mathbb{T}^2 ,¹⁰ hence the convenience of being a bit more sophisticated: the weight of \mathcal{L}_F can be written as the expansion of a one dimensional volume form on the vector space containing V . As V is exactly the tangent vector to the curves along which we integrate, we are led, as in [28], to consider functions on the Grassmannian made by one dimensional subspaces. However, in the simple case at hand, the construction in [28] can be considerably simplified. Namely, we can limit ourselves to considering the compact set $\Omega_* = \{(x, v) \in \mathbb{T}^2 \times \mathbb{R}^2 : \|v\| = 1, v \in \overline{C^s(x)}\}$. Moreover, since we have assumed that the stable distribution is orientable, then Ω_* is the disjoint union of two sets (corresponding to the two possible orientations). Let Ω be the connected component that contains the elements $(x, \widehat{V}(x))$. In addition, since we have also assumed that F preserves the orientation of the stable distribution, calling \mathbb{F} the projectivization of F_* we have $\Omega_0 = \mathbb{F}^{-1}(\Omega) \subset \Omega$.

Thus we have that $\mathbb{F} : \Omega_0 \subset \Omega \rightarrow \Omega$ is defined as

$$\begin{aligned}\mathbb{F}(x, v) &= (F(x), \|D_x F v\|^{-1} D_x F v), \\ \mathbb{F}^{-1}(x, v) &= (F^{-1}(x), \|D_x F^{-1} v\|^{-1} D_x F^{-1} v).\end{aligned}$$

Also note that

$$(2.7) \quad \mathbb{F}^{-1}(x, \widehat{V}(x)) = (F^{-1}(x), \widehat{V}(F^{-1}(x))).$$

Thus, if we define the natural extension $\phi_t(x, v) = (\phi_t(x), \|D_x \phi_t v\|^{-1} D_x \phi_t v)$, remembering (1.7), we have

$$\begin{aligned}\mathbb{F}^n(\phi_s(x, v)) &= (F^n(\phi_t(x)), \|D_x[F^n \circ \phi_t]v\|^{-1} D_x[F^n \circ \phi_t]v) \\ &= \left(\phi_{\tau_n(x, t)}(F^n(x)), \frac{D_{F^n(x)} \phi_{\tau_n(x, t)} D_x F^n v + V(\phi_{\tau_n(x, t)}(F^n(x)) \langle \nabla \tau_n(x, t), v \rangle)}{\|D_{F^n(x)} \phi_{\tau_n(x, t)} D_x F^n v + V(\phi_{\tau_n(x, t)}(F^n(x)) \langle \nabla \tau_n(x, t), v \rangle)\|} \right).\end{aligned}$$

⁹ Before [38] it was unclear if spaces with such properties existed at all. Nowadays there exists a profusion of possibilities. We use the ones stemming from [27, 28] because they seem particularly well suited for the task at hand, but any other possibility (e.g. [5, 6, 7]) should do.

¹⁰ Yet, one can obtain something uniformly proportional to the length, hence one could use the spaces detailed in [26], which would be well suited also for studying the geodesic versus horocyclic flows case. We prefer the following choice because it is a bit cleaner and it can be used for more general operators.

The above formula does not look very nice, however we will be only interested in integrations along the flow direction. Accordingly, by (1.3) and

$$(2.8) \quad D_x \phi_s(V(x)) = D_x \phi_s \left. \frac{d}{d\tau} \phi_\tau(x) \right|_{\tau=0} = \left. \frac{d}{d\tau} \phi_{s+\tau}(x) \right|_{\tau=0} = V(\phi_s(x)),$$

we have $D_{F^n(x)} \phi_{\tau_n(x,t)} D_x F^n V(x) = \nu_n(x) V(\phi_{\tau_n(x,t)}(F^n(x)))$. Hence, limited to $v = \widehat{V}(x)$, we recover the analogous of (1.7):

$$(2.9) \quad \mathbb{F}^n(\phi_s(x), \widehat{V}(x)) = \phi_{\tau_n(x,s)}(\mathbb{F}^n(x, \widehat{V}(x))).$$

We can now define the transfer operator associated to $\mathbb{F} : \mathcal{C}^0(\Omega_0, \mathbb{R}) \rightarrow \mathcal{C}^0(\Omega, \mathbb{R})$ as

$$(2.10) \quad \mathcal{L}_{\mathbb{F}} g(x, v) = g \circ \mathbb{F}^{-1}(x, v) \frac{\|D_x F^{-1} v\| \|V(x)\|}{\|V \circ F^{-1}(x)\|}.$$

The key observation is that $\pi \circ \mathbb{F}^{-1} = F^{-1} \circ \pi$, where we have introduced the projection $\pi(x, v) = x$. Hence, for each function $g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$, if we define $\mathbf{g} = \pi^* g := g \circ \pi$, then $\mathbf{g} \in \mathcal{C}^r(\Omega, \mathbb{R})$ and we have, for all $n \in \mathbb{N}$,

$$\mathcal{L}_{\mathbb{F}}^n g(x, \widehat{V}(x)) = \mathcal{L}_F^n g(x).$$

The above shows that understanding the properties of $\mathcal{L}_{\mathbb{F}}$ yields a control on \mathcal{L}_F . In addition, from definition (2.10) it is apparent that $\mathcal{L}_{\mathbb{F}}(\mathcal{C}^{r-1}(\Omega, \mathbb{R})) \subset \mathcal{C}^{r-1}(\Omega, \mathbb{R})$.¹¹ We have thus completely eliminated the above mentioned regularity problem.

Remark 2.5. *Note that \mathbb{F} is itself a uniformly hyperbolic map with the repeller $\{(x, v) \in \Omega : v = \widehat{V}(x)\}$ and it has an invariant splitting of the tangent space with two dimensional unstable distribution and one dimensional stable.*

Accordingly, we define, for each $\mathbf{g} \in \mathcal{C}^0(\Omega, \mathbb{R})$, $t \in \mathbb{R}_>$ and $x \in \mathbb{T}^2$, the new functional

$$(2.11) \quad \mathbb{H}_{x,t}(\mathbf{g}) = \int_0^t ds \mathbf{g}(\phi_s(x), \widehat{V}(\phi_s(x)))$$

and easily obtain the analogous of (2.6) for the operator $\mathcal{L}_{\mathbb{F}}$.

Lemma 2.6. *For each $\mathbf{g} \in \mathcal{C}^0(\Omega, \mathbb{R})$ and $g \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R})$, $n \in \mathbb{N}$, $t \in \mathbb{R}_>$ and $x \in \mathbb{T}^2$ we have*

$$\begin{aligned} \mathbb{H}_{x,t}(g \circ \pi) &= H_{x,t}(g) \\ \mathbb{H}_{x,t}(\mathbf{g}) &= \mathbb{H}_{F^n x, \tau_n(x,t)}(\mathcal{L}_{\mathbb{F}}^n \mathbf{g}). \end{aligned}$$

Proof. The proof of the first formula is obvious by the definition, the second follows by direct computation using (2.9):

$$\begin{aligned} \mathbb{H}_{x,t}(\mathbf{g}) &= \int_0^t ds \mathbf{g}(\phi_s(x), \widehat{V}(\phi_s(x))) = \int_0^t ds \mathbf{g} \circ \mathbb{F}^{-n} \circ \mathbb{F}^n \circ \phi_s(x, \widehat{V}(x)) \\ &= \int_0^{\tau_n(x,t)} ds_1 \nu_n(\mathbb{F}^{-n} \circ \phi_{s_1}(\mathbb{F}^n(x, \widehat{V}(x))))^{-1} \mathbf{g} \circ \mathbb{F}^{-n} \circ \phi_{s_1} \circ \mathbb{F}^n(x, \widehat{V}(x)) \\ &= \int_0^{\tau_n(x,t)} ds_1 (\mathcal{L}_{\mathbb{F}}^n \mathbf{g})(\phi_{s_1}(F^n(x), \widehat{V}(F^n(x)))) = \mathbb{H}_{F^n x, \tau_n(x,t)}(\mathcal{L}_{\mathbb{F}}^n \mathbf{g}). \end{aligned}$$

□

¹¹ Recall Definition 2.3-(ii).

The basic fact about the operator $\mathcal{L}_{\mathbb{F}}$ is that there exists Banach spaces $\mathcal{B}^{p,q}$,¹² detailed in Appendix C to which $\mathcal{L}_{\mathbb{F}}$ can be continuously extended.¹³ Moreover in Appendix C we prove the following result.

Proposition 2.7. *Let $F \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{T}^2)$ be an Anosov map. Let $p \in \mathbb{N}^*$ and $q \in \mathbb{R}$ such that $p + q \leq r$ and $q > 0$. Let $\rho = \exp(h_{\text{top}})$ where h_{top} is the topological entropy of F . Then the spectral radius of $\mathcal{L}_{\mathbb{F}}$ on $\mathcal{B}^{p,q}$ is ρ and its essential spectral radius is at most $\rho \lambda^{-\min\{p,q\}}$. In addition, ρ is a simple eigenvalue of $\mathcal{L}_{\mathbb{F}}$ and all the other eigenvalues are strictly smaller in norm.¹⁴*

Before being able to state precisely our first result we need another little bit of notation. Let $\{\mathbf{O}_{i,j}\}_{j=1}^{d_i} \subset (\mathcal{B}^{p,q})'$ be the elements of a base of the eigenspaces associated to the discrete eigenvalues $\{\rho_i\}_{i \geq 1}$, $|\rho_i| > \exp(h_{\text{top}}) \lambda^{-\min\{p,q\}}$, of $\mathcal{L}'_{\mathbb{F}}$ when acting on $(\mathcal{B}^{p,q})'$, $p + q \leq r - 1$.¹⁵ Since $\mathcal{C}^{p+q} \subset \mathcal{B}^{p,q}$, we have $(\mathcal{B}^{p,q})' \subset (\mathcal{C}^r)'$. Hence $\{\mathbf{O}_{i,j}\} \subset \mathcal{C}^r(\Omega, \mathbb{C})'$. We then define $\tilde{\mathbf{O}}_{i,j} = \pi_* \mathbf{O}_{i,j}$, clearly $\tilde{\mathbf{O}}_{i,j} \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{C})'$. Note that π_* is far from being invertible, so many different distributions could be mapped to the same one. Thus the dimension of the span of $\{\tilde{\mathbf{O}}_{i,j}\}_{j=1}^{d_i}$ will be, in general, smaller than d_j (see Appendix B.1 for an explicit example), let us call it $d_i \leq \tilde{d}_i$. Let $D_k = \sum_{i \leq k} d_i$. For convenience, let us relabel our distributions $\{O_i\}$, by $O_i = \tilde{O}_{k,l}$ for $i \in [D_k + 1, D_{k+1}]$ and $l = i - D_k$.

Theorem 2.8. *Provided r is large enough,¹⁶ there exists N_1 such that the distributions (obstructions) $\{O_i\}_{i=1}^{N_1} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{C})'$ have the following properties. For each $k \leq N_1$, let $\mathbb{V}_k = \{g \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{C}) : O_j(g) = 0 \ \forall j < k ; O_k(g) \neq 0\}$. Then there exists $C > 0$ such that, for all for all $x \in \mathbb{T}^2$ and $g \in \mathbb{V}_k$, we have*

$$|H_{x,t}(g)| \leq \begin{cases} C t^{\alpha_k} (\ln t)^{b_k} \|g\|_{\mathcal{C}^r} & \text{if } \alpha_k > 0 \\ C (\ln t)^{b_k+1} \|g\|_{\mathcal{C}^r} & \text{if } \alpha_k = 0, \end{cases}$$

where $i \in (D_{k-1}, D_k]$, $\alpha_k = \frac{\ln |\rho_k|}{h_{\text{top}}}$ and $b_k \in \{0, \dots, d_k\}$. Also $\alpha_1 = 1$, $b_1 = 0$ and $\alpha_{N_1} = b_{N_1} = 0$.

The above Theorem will be proven in Section 3.

Remark 2.9. *Note that in Theorem 2.8 it could happen $N_1 = 1$, this is indeed the case in the linear case (see Appendix B.1). In such a case the result is less interesting, yet it always gives a relevant information.*

Remark 2.10. *A natural question that arises is how to obtain a more explicit identification of the above mentioned distributions. In particular, the analogy with the situations studied by Flaminio-Forni would suggest $(\phi_t)_* O_j = O_j$, that is the distributions are invariant for the flow. We expect this to be true but the proof is not so obvious: due to the low regularity of the flow, $(\phi_t)_* O_j$ might be a distribution*

¹² These are more general with respect to the previously mentioned ones. We give them the same name to simplify notation and since no confusion can arise.

¹³ By a slight abuse of notations we will still call $\mathcal{L}_{\mathbb{F}}$ such an extension.

¹⁴ The conditions on p, q are not optimal. The lack of optimality begin due to the fact that we require $p \in \mathbb{N}$. See [6], and reference therein, for different approaches that remove such a constraint.

¹⁵ Remark that the compact part of the spectrum of $\mathcal{L}_{\mathbb{F}}$ and $\mathcal{L}'_{\mathbb{F}}$ coincide (see [31, Remark 6.23]).

¹⁶ For example, $e^{h_{\text{top}}} \lambda^{-r/2} < 1$ suffices. We refrain from giving a more precise characterization of the minimal r since, in the present context, it is not very relevant.

of low regularity and hence not belonging to the spaces we are considering. This problem could be bypassed by showing that the norm of O_j is bounded by the functionals $H_{x,t}$. This is certainly true for some O_j , but it is not obvious how to prove it in general. We therefore limit ourself to the discussion of O_1 .

Lemma 2.11. *The distribution O_1 is proportional to the unique invariant measure μ of ϕ_t .*

Proof. By Proposition 2.7 it follows that, for all $g \geq 0$,

$$0 \leq \lim_{n \rightarrow \infty} \rho^{-n} \mathcal{L}_{\mathbb{F}}^n(g) = h_1 O_1(g).$$

Accordingly, O_1 is a positive distribution, and hence a measure, thus also O_1 is a measure. By the ergodic theorem $H_{x,t}(g)$ grows proportional to t unless $g \in \mathbb{V}_0 = \{g : \mu(g) = 0\}$. By Theorem 2.8 it follows that $\text{Ker}(O_1) \subset \mathbb{V}_0$. On the other hand the kernel of O_1 must be a codimension one closed subspace, hence $\text{Ker}(O_1) = \mathbb{V}_0$. It follows that the two measures must be proportional.¹⁷ \square

The next step is to study, in the case $O_i(g) = 0$ for all $i \in \{1, \dots, N_1\}$, the regularity of the coboundary. As already mentioned (see Remark 1.5) it is natural to consider only $r_i \leq 1 + \alpha$. To study exactly the Hölder regularity would entail either to use a more complex Banach space or an interpolating argument. As already mentioned, in the spirit of giving ideas rather than a complete theory, we content ourselves with considering Lipschitz regularity. To do so we have only to consider the derivative of \overline{H}_T with respect to x . To study the growth of such a derivative, it is necessary to introduce a new adapted transfer operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ which is now defined on one forms (see 4.7 for the precise definition) and acts on different Banach spaces $\widehat{\mathcal{B}}^{p,q}$ (see Appendix D). The Banach spaces $\widehat{\mathcal{B}}^{p,q}$ are a bit more complex than the $\mathcal{B}^{p,q}$ used in Theorem 2.8 insofar they are really spaces of currents rather than distributions (one has to think of dg , rather than g , as an element of the Banach space). Apart from this, the proof of our next result, to be found in Section 4, follows the same logic of the first proof.

Theorem 2.12. *Provided r is large enough,¹⁸ there exist distributions $\{O_i\}_{i=N_1+1}^{N_2} \subset \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})'$ such that if $O_i(g) = 0$ for all $i \in \{1, \dots, N_2\}$, then g is a Lipschitz coboundary. In addition, there exists Banach spaces $\widehat{\mathcal{B}}^{p,q}$ and a transfer operator $\widehat{\mathcal{L}}_{\mathbb{F}}$, determined by the map F , such that the $\{O_i\}_{i=N_1+1}^{N_2}$ are determined, as in Theorem 2.8, by a base of the eigenspaces associated to the discrete eigenvalues $\{\rho_i\}$ of $\widehat{\mathcal{L}}_{\mathbb{F}}$ when acting on $\widehat{\mathcal{B}}^{p,q}$ for appropriate p, q , $p + q \leq r - 2$.*

The rest of the paper is devoted to the proof of the above claims. Last we would like to conclude with the following

Conjecture 2.13. *The natural analogous of Theorems 2.8 and 2.12 hold in the case of horocycle flow on a surface of variable strictly negative curvature, were the renormalizing dynamics is the geodesic flow, with the only modification of having an infinite countable family of obstructions.*

¹⁷ Remark that this implies that O_1 is invariant for the flow ϕ_t . In fact, by using judiciously (2.9) one could have proven directly that O_1 is invariant for ϕ_t . It is possible that such a proof would work also for eigendistributions with eigenvalues with modulus sufficiently close to one. Yet, for smaller eigenvalues the aforementioned regularity problems seem to kick in.

¹⁸ Here r needs to be much larger than in the previous Theorem. A precise estimate is implicit in the proof, but the reader may be better off assuming $F \in \mathcal{C}^\infty$ and not worrying about this issue.

Remark 2.14. *The difference between finitely many and countably many obstructions, comes from the different spectrum of the transfer operators for maps and flows. In the former, the discrete spectrum is always finite. In the latter, one has a one parameter families of operators and it is then more natural to look at the spectrum of the generator. It turns out that such a spectrum is discrete on the right of a vertical line whose location depends on the flow regularity. Yet, it can have countably many eigenvalues (as the laplacian on hyperbolic spaces), hence the countably many obstructions (see [9, 10] for more details).*

3. GROWTH OF THE ERGODIC AVERAGE

Before providing the proof of Theorem 2.8 there is one further, and luckily last, conceptual obstacle preventing the naive implementation our strategy: the functionals $\mathbb{H}_{x,t}$ are, in general, not continuous (let alone uniformly continuous) on the spaces $\mathcal{B}^{p,q}$ that are detailed in Appendix C.¹⁹ In fact, it is possible to introduce different Banach spaces on which the transfer operator is quasi-compact and the functionals $\mathbb{H}_{x,t}$ are continuous (this are spaces developed to handle piecewise smooth dynamics such as [12, 3, 13, 4]) but the essential spectral radius of our transfer operators on such spaces would always be rather large. Hence we would be able to obtain in this way, at best, only the very firsts among the relevant distributions we are seeking, whereby nullifying the appeal of our approach. To circumvent this last problem we introduce, for each $x \in \mathbb{T}^2$ and $\varphi \in L^\infty(\mathbb{R}_+, \mathbb{R})$, the new “mollified” functional

$$(3.1) \quad \mathbb{H}_{x,\varphi}(\mathbf{g}) = \int_{\mathbb{R}} \varphi(t) \cdot \mathbf{g} \circ \phi_t(x, \widehat{V}(x)) dt.$$

It is proven in Appendix C that $\mathbb{H}_{x,\varphi} \in (\mathcal{B}^{p,q})'$ provided $\varphi \in \mathcal{C}_0^{p+q}(\mathbb{R}, \mathbb{R})$.

3.1. Proof of our first main result.

Our key claim is that the functionals (3.1) suffice for our purposes. To be more precise let us fix $t > 0$ and define the sets $\mathcal{D}_{r,C}^s = \{\varphi \in \mathcal{C}^r([0, t], \mathbb{R}_+) : \|\varphi\|_{\mathcal{C}^r} \leq C\}$ and $\mathcal{D}_{r,C} = \{\varphi \in \mathcal{C}_0^r([0, t], \mathbb{R}_+) : \|\varphi\|_{\mathcal{C}^r} \leq C\}$, note that such sets are locally compact in $\mathcal{C}^{r-1}([0, t], \mathbb{R})$ and $\mathcal{C}_0^{r-1}([0, t], \mathbb{R})$, respectively.²⁰

Lemma 3.1. *There exists $C_* > 0$ such that, for each $n \in \mathbb{N}$, $t \in \mathbb{R}_+$, $x \in \mathbb{T}^2$ and $\mathbf{g} \in \mathcal{C}^{r-1}(\Omega, \mathbb{R})$, there exists $K \in \mathbb{N}$, $\{n_i^\pm\}_{i=1}^K \subset \mathbb{N}$, $n_K^\pm = 0$, $n_i^\pm \geq n_{i+1}^\pm$, $n_i^- + n_i^+ > n_{i+1}^- + n_{i+1}^+$, and $\{\varphi_i^\pm\}_{i=1}^K \subset \mathcal{D}_{r,C_*}$, $\{\varphi^\pm\} \subset \mathcal{D}_{r,C_*}^s$ such that*

$$\mathbb{H}_{x,t}(\mathbf{g}) = \sum_{\sigma \in \{+, -\}} \sum_{i=1}^K \mathbb{H}_{F^{n_i^\sigma}(x), \varphi_i^\sigma}(\mathcal{L}_{\mathbb{F}}^{n_i^\sigma} \mathbf{g}) + \mathbb{H}_{x,\varphi^\sigma}(\mathbf{g}).$$

Moreover, $\max\{|\text{supp } \varphi^\pm|, |\text{supp } \varphi_i^\pm|\} \leq 1$.

Finally $\varphi_1^- = \varphi_1^+$ and $n_1^\pm = n_t$ where $n_t = \inf\{n \in \mathbb{N} : \tau_n(x, t) < 1\}$ satisfies the

¹⁹ The problem comes from the boundary in the domain of the integral defining them. There, in some sense, the integrand jumps to zero and cannot be considered smooth in any effective manner.

²⁰ Up to now the exact definition of the \mathcal{C}^r norms was irrelevant, now instead it does matter. We make the choice $\|\varphi\|_{\mathcal{C}^r} = \sum_{k=0}^r 2^{r-k} \|\varphi^{(k)}\|_{L^\infty}$. It is well known that with such a norm \mathcal{C}^r is a Banach algebra. Also, as usual, for a \mathcal{C}^r function on a closed set, we mean that there exists an extension on some larger open set.

bounds

$$(3.2) \quad \frac{\ln t}{h_{\text{top}}} - C_{\#} \leq n_t \leq \frac{\ln t}{h_{\text{top}}} + C_{\#}.$$

Before proving Lemma 3.1, let us use it and prove our first main result.

Proof of Theorem 2.8. By Proposition 2.7 we have

$$(3.3) \quad \mathcal{L}_{\mathbb{F}} = \sum_{j=0}^m (\rho_j \Pi_j + Q_j) + R_{p,q}$$

where m is a finite number, ρ_j , $|\rho_{j+1}| \leq |\rho_j| \leq e^{h_{\text{top}}}$, are complex eigenvalues of $\mathcal{L}_{\mathbb{F}}$, Π_j are finite rank projectors, Q_j are nilpotent operators. That is, there exists $\{d_j\}_{j=1}^m$ such that $Q_j^{d_j} = 0$ and, if $d_j > 1$, then $Q_j^{d_j-1} \neq 0$. Finally, $R_{p,q}$ is a linear operator with spectral radius at most $e^{\beta_{\text{ess}}} = \lambda^{-\min(p,q)} e^{h_{\text{top}}}$. In addition, $\Pi_j R_{p,q} = R_{p,q} \Pi_j = Q_j R_{p,q} = R_{p,q} Q_j = 0$. Moreover, for each $i \neq j$, $\Pi_i \Pi_j = \Pi_j \Pi_i = 0$, $\Pi_i Q_j = Q_i \Pi_j = Q_j$, $Q_i Q_j = Q_j Q_i = 0$ and $\Pi_i Q_i = Q_i \Pi_i$. In other words the operator $\mathcal{L}_{\mathbb{F}}$ is quasi compact and it has a spectral decomposition in Jordan Block of size d_j plus a non compact part of small spectral radius. Note as well that $d_1 = 1$, $Q_1 = 0$ and Π_1 is a one dimensional projection corresponding to the eigenvalue $e^{h_{\text{top}}}$ which is the only eigenvalue of modulus $e^{h_{\text{top}}}$. Finally, set

$$\alpha_j = \frac{\ln |\rho_j|}{h_{\text{top}}}; \quad \tilde{N}_1 = \sum_{j=1}^m d_j.$$

If r is large enough, we can choose p, q and m such that $\beta_{\text{ess}} < 0$ and $|\rho_j| \geq 1$ for all $j \leq m$. We discuss first the case $|\rho_j| > 1$, i.e. $\alpha_j > 0$.

Then, setting $\mathbf{g} = g \circ \pi$, Lemmata 2.6, 3.1 and C.4 imply that, for all $\epsilon > 0$,²¹

$$(3.4) \quad \begin{aligned} |H_{x,t}(g)| &\leq C_{\#} \|g\|_{L^{\infty}} + C_{\#} \sum_{n=1}^{n_t} \left[\sum_{j=0}^m \|(\rho_j \Pi_j + Q_j)^n \mathbf{g}\|_{p,q} + \|R_{p,q}^n \mathbf{g}\|_{p,q} \right] \\ &= C_{\#} \|g\|_{L^{\infty}} + C_{\#} \sum_{n=1}^{n_t} \left[\sum_{j=0}^m \rho_j^n n^{d_j} \|\Pi_j \mathbf{g}\|_{p,q} + \|\mathbf{g}\|_{p,q} \right]. \end{aligned}$$

Thus we obtain

$$(3.5) \quad |H_{x,t}(g)| \leq C_{\#} \|g\|_{C^r} + C_{\#} \sum_{n=1}^{n_t} \sum_{j=0}^m t^{\alpha_j} (\ln t)^{d_j} \|\Pi_j \mathbf{g}\|_{p,q}.$$

To conclude note that, since the $\Pi_j = \sum_{i=1}^{d_j} h_{j,i} \otimes \mathbf{O}_{j,i}$ with $h_{j,i} \in \mathcal{B}^{p,q}$ and $\mathbf{O}_{j,i} \in (\mathcal{B}^{p,q})' \subset (C^r(\Omega, \mathbb{R}))'$. Finally, since $\pi^* : C^r(\mathbb{T}^2, \mathbb{R}) \subset C^r(\Omega, \mathbb{R})$, we have that $\tilde{\mathbf{O}}_{j,i} := \pi_* \mathbf{O}_{i,j} \in (C^r(\mathbb{T}^2, \mathbb{R}))'$, and $\mathbf{O}_{i,j}(g) = \tilde{\mathbf{O}}_{i,j}(g)$. Note that it might happen $\pi_* \mathbf{O}_{i,j} = \pi_* \mathbf{O}_{i',j'}$ (see Appendix B.1). Let $N_1 \leq \tilde{N}_1$ to be the cardinality of the set $\{\pi_* \mathbf{O}_{i,j}\}$. This finally yields the wanted upper bound

$$(3.6) \quad |H_{x,t}(g)| \leq C_{\#} \|g\|_{C^r} + C_{\#} \sum_{n=1}^{n_t} \sum_{i=0}^m t^{\alpha_i} (\ln t)^{d_i} \sum_{j=1}^{d_i} |\mathbf{O}_{i,j}(g)|.$$

²¹ Note that the n_i^{\pm} in Lemma 3.1 cannot be more than n_t , hence $K \leq n_t$.

If there exists $m_0 < m$ such that $\alpha_j = 0$ for all $j > m_0$, hence $|\rho_j| = 1$, the argument can be carried out exactly in the same way, a part that the sum in the second line of (3.4) is not longer a geometric sum, and hence an extra power appears in the related logarithms in the analogous of equation (3.5). \square

3.2. Decomposition in proper functionals.

This section is devoted to showing that the functionals $H_{x,t}$ can be written in terms of well behaved functionals plus a bounded error.

Proof of Lemma 3.1. Fix $x \in \mathbb{T}^2$ and $t \in \mathbb{R}_{>}$. By definition $\tau_{n_t}(x, t) \in (\Lambda^{-1}, 1)$ for some fixed $\Lambda > 1$.

Let $\delta \in (0, \Lambda^{-1}/4)$ small and $C_* > 0$ large enough to be fixed later. We can now fix $n_1 = n_t$. Note that the claimed bound on n_t follows directly by [26, Lemma C.3]. Next, chose $\psi \in \mathcal{D}_{r, C_*/2}$ such that $\text{supp } \psi \subset (\delta, \tau_{n_1} - \delta)$, $\psi|_{[2\delta, \tau_{n_1} - 2\delta]} = 1$. Set $\psi^- = (1 - \psi)\mathbb{1}_{[0, \tau_{n_1}/2]}$, $\psi^+ = (1 - \psi)\mathbb{1}_{[\tau_{n_1}/2, \tau_{n_1}]}$. Then $\psi^\pm \in \mathcal{D}_{r, C_*}^s$ and we can use Lemma 2.6 to write

$$\begin{aligned} \mathbb{H}_{x,t}(\mathbf{g}) &= \mathbb{H}_{F^{n_1}(x), \tau_{n_1}(x,t)}(\mathcal{L}_{\mathbb{F}}^{n_1} \mathbf{g}) \\ &= \mathbb{H}_{F^{n_1}(x), \psi^-}(\mathcal{L}_{\mathbb{F}}^{n_1} \mathbf{g}) + \mathbb{H}_{F^{n_1}(x), \psi}(\mathcal{L}_{\mathbb{F}}^{n_1} \mathbf{g}) + \mathbb{H}_{F^{n_1}(x), \psi^+}(\mathcal{L}_{\mathbb{F}}^{n_1} \mathbf{g}). \end{aligned}$$

We are happy with the middle term which, by Lemma C.4, is a continuous functional of $\mathcal{L}_{\mathbb{F}}^{n_1} \mathbf{g}$, not so the other two terms. We have thus to take care of them. A computation analogous to the one done in Lemma 2.6 yields, for each $n \in \mathbb{N}$,

$$(3.7) \quad \mathbb{H}_{x, \varphi \circ \tau_n(x, \cdot)}(\mathbf{g}) = \mathbb{H}_{F^n(x), \varphi}(\mathcal{L}_{\mathbb{F}}^n \mathbf{g}).$$

We will use the above to prove inductively the formula

$$\begin{aligned} (3.8) \quad \mathbb{H}_{x,t}(\mathbf{g}) &= \mathbb{H}_{F^{n_k}^-(x), \psi_k^-}(\mathcal{L}_{\mathbb{F}}^{n_k^-} \mathbf{g}) + \mathbb{H}_{F^{n_k}^+(x), \psi_k^+}(\mathcal{L}_{\mathbb{F}}^{n_k^+} \mathbf{g}) \\ &\quad + \sum_{\sigma \in \{+, -\}} \sum_{i=1}^k \mathbb{H}_{F^{n_i}^\sigma(x), \varphi_i^\sigma}(\mathcal{L}_{\mathbb{F}}^{n_i^\sigma} \mathbf{g}) \end{aligned}$$

where $\psi_1^\pm = \psi^\pm$, $\varphi_1^\pm = \frac{1}{2}\psi$, $n_1^\pm = n_1$, $\psi_k^\pm \in \mathcal{D}_{r, C_*}^s$, $\{\varphi_i^\pm\} \subset \mathcal{D}_{r, C_*}$, $\|\psi_k^\pm\|_{L^\infty} \leq 1$, $\|\varphi_k^\pm\|_{L^\infty} \leq 1$, $(b_k^\pm, b_k^\pm \mp \delta) \subset \text{supp } \psi_k^\pm \subset (b_k^\pm, b_k^\pm \mp 2\delta)$, $\text{supp } \varphi_k^\pm \subset (b_k^\pm, b_k^\pm \mp 1)$, $b_k^- = 0$ and $b_k^+ \in [0, \Lambda^{n_k^+}]$, $b_1^+ = t$.

Let us consider the first term on the right hand side of the first line of (3.8) (the second one can be treated in total analogy). Let $\text{supp}(\psi_k^-) = [0, a_k]$ and define $m+1 = \inf\{n \in \mathbb{N} : \tau_n^{-1}(F^{n_k}(x), a_k) \geq 1\}$. Note that, by construction, there exists a fixed $\overline{m} \in \mathbb{N}$ such that $m \geq \overline{m}$, also \overline{m} can be made large by choosing δ small. Hence $\widehat{\psi}_k^-(s) = \psi_k^- \circ \tau_m(F^{n_k}(x), s)$ is supported in the interval $[0, 1)$ and the support contains $[0, \Lambda^{-1}]$.

Next, we need an estimate on the norm of $\widehat{\psi}_k^-$. We state it in a sub-lemma so the reader can easily choose to skip the, direct but rather tedious, proof.

Sub-Lemma 3.2. *Provided we choose δ small and C_* large enough, we have*

$$\widehat{\psi}_k^- \in \mathcal{D}_{r, C_*/2},$$

where $n_{k+1}^- = n_k^- - m$.

Proof. First of all $\|\psi_k^-\|_{L^\infty} \leq 1$, and²²

$$\begin{aligned} \|\widehat{\psi}_k^-\|_{C^r} &\leq \sum_{j=0}^r 2^{r-j} \|\psi_k^-\|_{C^j} \|\tilde{\nu}_{z_k, m}\|_{C^{r-1}} \|\tilde{\nu}_{z_k, m}\|_{C^{r-2}} \cdots \|\tilde{\nu}_{z_k, m}\|_{C^{r-j}} \\ (3.9) \quad &\leq 2^r + C_* \sum_{j=1}^r \|\tilde{\nu}_{z_k, m}\|_{C^{r-1}} \|\tilde{\nu}_{z_k, m}\|_{C^{r-2}} \cdots \|\tilde{\nu}_{z_k, m}\|_{C^{r-j}} \end{aligned}$$

where $z_k = F^{n_k^-}(x)$ and, for each $j \in \mathbb{N}$ and $z \in \mathbb{T}^2$, $\tilde{\nu}_{z, m}(s) = \nu_m(\phi_s(z))$, where ν_m is defined in (1.3). Note that, although ν_m is, in general, only $C^{1+\alpha}$, by (1.4) it follows $\tilde{\nu}_{z, m} \in C^{r-1}$ and hence, for all $z \in \mathbb{T}^2$, $\langle V, \nabla \tilde{\nu}_z \rangle \circ \phi(\cdot) \in C^{r-2}$. We can thus continue and compute

$$\begin{aligned} \frac{d}{ds} \tilde{\nu}_{z_k, m}(s) &= \tilde{\nu}_{z_k, m}(s) \sum_{l=0}^{m-1} \frac{\langle \nabla \nu_1(F^l \circ \phi_s(z_k)), V(F^l \circ \phi_s(z_k)) \rangle}{\nu_1(F^l \circ \phi_s(z_k))} \tilde{\nu}_{z_k, l}(s) \\ &= \tilde{\nu}_{z_k, m}(s) \sum_{l=0}^{m-1} \left[\frac{\langle V, \nabla \nu_1 \rangle}{\nu_1} \right] \circ \phi_{\tau_l(z_k, s)}(F^l(z_k)) \tilde{\nu}_{z_k, l}(s). \end{aligned}$$

The above, by induction, implies that there exist increasing constants $A_q \geq 1$ such that $\|\tilde{\nu}_{z_k, m}\|_{C^q} \leq A_q \|\tilde{\nu}_{z_k, m}\|_{C^0}$. Indeed, $\left[\frac{\langle V, \nabla \nu_1 \rangle}{\nu_1} \right] \circ \phi \in C^{r-1}$, and

$$\begin{aligned} \left\| \left[\frac{\langle V, \nabla \nu_1 \rangle}{\nu_1} \right] \circ \phi_{\tau_l(z_k, \cdot)}(F^l(z_k)) \right\|_{C^q} &\leq \sum_{i=0}^q 2^{q-i} \left\| \left[\frac{\langle V, \nabla \nu_1 \rangle}{\nu_1} \right] \circ \phi(F^l(z)) \right\|_{C^q} \|\tilde{\nu}_{F^l(z_k), l}\|_{C^{q-1}}^i \\ &\leq C_\# \sum_{i=0}^q A_{q-1}^i \lambda^{-il} \leq C_\# A_{q-1}^q. \end{aligned}$$

Thus,

$$\begin{aligned} \|\tilde{\nu}_{z_k, m}\|_{C^{q+1}} &= 2^q \|\tilde{\nu}_{z_k, m}\|_{C^0} + \left\| \frac{d}{ds} \tilde{\nu}_{z_k, m} \right\|_{C^q} \\ &\leq 2^q \|\tilde{\nu}_{z_k, m}\|_{C^0} + \|\tilde{\nu}_{z_k, m}\|_{C^q} \sum_{l=0}^{m-1} C_\# A_{q-1}^q A_{q-1} \lambda^{-l} \\ &\leq \left[2^q + A_q C_\# A_{q-1}^{q+1} \right] \|\tilde{\nu}_{z_k, m}\|_{C^0} =: A_{q+1} \|\tilde{\nu}_{z_k, m}\|_{C^0}. \end{aligned}$$

We did not try to optimize the above computation since the only relevant point is that the A_q do not depend on m . Accordingly, if we choose δ small (and hence m large) enough, we have $\|\tilde{\nu}_{z_k, m}\|_{C^q} \leq \frac{1}{4}$ for all $q \leq r-1$. Using the above fact in (3.9) yields

$$\|\widehat{\psi}_k^-\|_{C^r} \leq 2^r + C_* \sum_{j=1}^r 4^{-j} = 2^r + \frac{1}{3} C_*$$

which implies the Lemma provided we choose C_* large. \square

By (3.7), we have

$$\mathbb{H}_{F^{n_k}^-(x), \psi_k^-}(\mathcal{L}_{\mathbb{F}}^{n_k} g) = \mathbb{H}_{F^{n_{k+1}}^-(x), \widehat{\psi}_k^-}(\mathcal{L}_{\mathbb{F}}^{n_{k+1}} g).$$

²² Here we use the formula $\|f \circ g\|_{C^r} \leq \sum_{k=0}^r 2^{r-k} \|f\|_{C^k} \|Dg\|_{C^{r-1}} \|Dg\|_{C^{r-2}} \cdots \|Dg\|_{C^{r-k}}$, that can be verified by induction.

Again we can write $\psi_{k+1}^- = (1 - \psi)\widehat{\psi}_k^- \mathbf{1}_{[0,2\delta]}$ and $\varphi_{k+1}^- = \widehat{\psi}_k^- - \psi_{k+1}^-$. Then,

$$\sup\{\|\psi_{k+1}^-\|_{C^r([0,1],\mathbb{R})}, \|\varphi_{k+1}^-\|_{C_0^r(\mathbb{R},\mathbb{R})}\} \leq C_*$$

and $[0, \delta] \subset \text{supp } \psi_{k+1}^- \subset [0, 2\delta]$. Accordingly

$$\mathbb{H}_{F^{n_k}^-(x), \psi_k^-}(\mathcal{L}_{\mathbb{F}}^{n_k^-} \mathbf{g}) = \mathbb{H}_{F^{n_{k+1}}^-(x), \psi_{k+1}^-}(\mathcal{L}_{\mathbb{F}}^{n_{k+1}^-} \mathbf{g}) + \mathbb{H}_{F^{n_{k+1}}^-(x), \varphi_{k+1}^-}(\mathcal{L}_{\mathbb{F}}^{n_{k+1}^-} \mathbf{g}).$$

The Lemma is thus proven by taking $k = K$, so that $n_K^\pm = 0$.²³ \square

4. COBOUNDARY REGULARITY

We start this section by proving several claims stated in the introduction and setting up some notation. Then we prove our main results concerning coboundary regularity.

4.1. Parabolic.

In the introduction we called the flow ϕ_t a *parabolic* dynamics, but no evidence was provided for this name. It is now time to substantiate such an assertion.

Lemma 4.1. *There exists $C, \beta > 0$ such that, for all $x \in \mathbb{T}^2$ and $t \in \mathbb{R}_+$, letting $\xi(s) = D_x \phi_s$, we have*

$$\|\xi\|_{C^{r-1}((0,t), GL(2,\mathbb{R}))} \leq C t^\beta.$$

Proof. It turns out to be convenient to define $V^\perp(x)$ as the perpendicular vector to $V(x)$ such that $\|V^\perp(x)\| = \|V(x)\|^{-1}$. In this way we can use $\{V(x), V^\perp(x)\}$ as basis of the tangent space at x , and the changes of variable are uniformly bounded, with determinant one and C^r in the flow direction. In such coordinates we have

$$(4.1) \quad D_x \phi_t = \begin{pmatrix} 1 & a(x, t) \\ 0 & b(x, t) \end{pmatrix}.$$

To have a more precise understanding of the above matrix elements, we have to use the knowledge that the dynamics is renormalizable. To start with we must differentiate (1.7):

$$\begin{aligned} D_{\phi_t(x)} F^n \cdot D_x \phi_t &= D_{F^n(x)} \phi_{\tau_n(x,t)} \cdot D_x F^n + V(\phi_{\tau_n(x,t)}(F^n(x))) \otimes \nabla \tau_n(x, t) \\ &= D_{F^n(x)} \phi_{\tau_n(x,t)} \cdot D_x F^n [\mathbf{1} + \nu_n(x)^{-1} V(x) \otimes \nabla \tau_n(x, t)], \end{aligned}$$

where we have used (1.3) and (2.8).

Hence, setting $A_{x,t,n} = \mathbf{1} + \nu_n(x)^{-1} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x, t))$, we have

$$(4.2) \quad D_x \phi_t = D_{\phi_{\tau_n(x,t)} \circ F^n(x)} F^{-n} \cdot D_{F^n(x)} \phi_{\tau_n(x,t)} \cdot D_x F^n \cdot A_{x, \tau_n(x,t), n}.$$

By equations (4.1) and (4.2) we have, for each $n \in \mathbb{N}$,

$$b(x, t) = \langle V^\perp(\phi_t(x)), D_{\phi_{\tau_n(x,t)} \circ F^n(x)} F^{-n} \cdot D_{F^n(x)} \phi_{\tau_n(x,t)} \cdot D_x F^n V^\perp(x) \rangle.$$

We then choose n so that $\tau_n \in [\Lambda^{-1}, 1]$, hence n is proportional to $\ln t$. By compactness it follows that $\|D_{F^n(x)} \phi_{\tau_n(x,t)}\| \leq C_\#$. Hence there exists $\beta_0 > 0$ such that

$$\sup_{x \in \mathbb{T}^2} |b(x, t)| \leq C_\# t^{\beta_0}.$$

²³ If more steps are needed on one side, say the plus side, one can simply set $n_{k+1}^- = n_k^-$, $\psi_{k+1}^- = \psi_k^-$ and $\varphi_k^- = 0$ for all the extra steps.

On the other hand, by the semigroup property, for each $m \in \mathbb{N}$,

$$D_x \phi_m = \prod_{i=0}^{m-1} \begin{pmatrix} 1 & a(\phi_i(x), 1) \\ 0 & b(\phi_i(x), 1) \end{pmatrix} = \begin{pmatrix} 1 & \sum_{j=0}^{m-1} a(\phi_j(x), 1)b(x, m-j) \\ 0 & b(x, m) \end{pmatrix}.$$

Since, again by compactness, $|a(x, 1)| \leq C_\#$, it follows

$$|a(x, m)| \leq C_\# \sum_{j=0}^{m-1} |b(x, m-j)| \leq C_\# \sum_{j=0}^{m-1} j^{\beta_0} \leq C_\# m^{\beta_0+1}.$$

Hence

$$\|\xi\|_{\mathcal{C}^0((0,t), GL(2, \mathbb{R}))} \leq C_\# t^\beta$$

with $\beta = \beta_0 + 1$.

To estimate the derivative notice that $\dot{\xi}(s) = D_{\phi_s(x)} V \xi(s)$. To understand the regularity of the above equation, recall that the stable foliation can be expressed in local coordinates by $(x_1, G(x_1, x_2))$, where $G(\cdot, x_2) \in \mathcal{C}^r$, $G(0, x_2) = x_2$, so that $\{(x_1, G(x_1, x_2))\}_{x_1 \in \mathbb{R}}$ is the leaf through the point $x = (x_1, x_2)$, and $(1, \partial_{x_1} G(x)) = V(x)$. It is known that, in such coordinates, $\partial_{x_2} G(\cdot, x_2) \in \mathcal{C}^{r-1}$ uniformly, see [29] and references therein. Then, by Schwarz Theorem, it follows that $\partial_{x_2} \partial_{x_1} G(\cdot, x_2) \in \mathcal{C}^{r-2}$. Hence, $DV \circ \phi_t$ is a \mathcal{C}^{r-2} function of t , with uniformly bounded norm. Accordingly, for each $k \in \{0, \dots, r-2\}$,

$$\begin{aligned} \|\xi\|_{\mathcal{C}^{k+1}((0,t), GL(2, \mathbb{R}))} &\leq \|\dot{\xi}\|_{\mathcal{C}^k((0,t), GL(2, \mathbb{R}))} + 2^{k+1} \|\xi\|_{\mathcal{C}^0((0,t), GL(2, \mathbb{R}))} \\ &\leq C_\# \|\xi\|_{\mathcal{C}^k((0,t), GL(2, \mathbb{R}))}, \end{aligned}$$

from which the Lemma readily follows. \square

4.2. Measurable coboundary.

In Section 3.1 we have seen that if g belongs to the kernel of enough O_i , then the $H_{x,t}(g)$ are all uniformly bounded. In the introduction we claimed that the same is true for $\overline{H}_{x,t}(g)$ now is the time to prove it.

Proof of Lemma 1.4. Setting, as before, $\mathbf{g} = g \circ \pi$, by equations (1.9), (3.1), (3.7) and arguing as in Lemma 2.6 we have

$$\overline{H}_{x,t}(g) = -\mathbb{H}_{x, \chi \circ \tau_n(x, \cdot)}(\mathbf{g}) = -\mathbb{H}_{F^n(x), \chi}(\mathcal{L}_{\mathbb{F}}^n \mathbf{g}).$$

From the hypotheses $\mathcal{L}_{\mathbb{F}}^n \mathbf{g}$ is uniformly bounded in the $\|\cdot\|_{p,q}$ norms. Thus, by Lemma C.4, also $\mathbb{H}_{F^n(x), \chi}(\mathcal{L}_{\mathbb{F}}^n \mathbf{g})$ is uniformly bounded and hence the same is true for $\overline{H}_{x,t}(g)$.

To prove the second statement of the Lemma, observe that

$$\begin{aligned}
\langle V(x), \nabla \overline{H}_T(g)(x) \rangle &= - \int_0^T dt \chi \circ \tau_{n_T}(x, t) \langle D_x \phi_t V(x), (\nabla g) \circ \phi_t(x) \rangle \\
&\quad - \int_0^T dt \chi' \circ \tau_{n_T}(x, t) \left[\int_0^s \langle D_x \phi_s V(x), (\nabla \nu_{n_T}) \circ \phi_s(x) \rangle ds \right] g \circ \phi_t(x) dt \\
&= - \int_0^T dt \chi \circ \tau_{n_T}(x, t) \langle V(\phi_t(x)), (\nabla g) \circ \phi_t(x) \rangle \\
&\quad - \int_0^T dt \chi' \circ \tau_{n_T}(x, t) \left[\int_0^s \langle V(\phi_s(x)), (\nabla \nu_{n_T}) \circ \phi_s(x) \rangle ds \right] g \circ \phi_t(x) dt \\
(4.3) \quad &= - \int_0^T dt \chi \circ \tau_{n_T}(x, t) \left(\frac{d}{dt} g \circ \phi_t(x) \right) \\
&\quad - \int_0^T dt \chi' \circ \tau_{n_T}(x, t) [\nu_{n_T} \circ \phi_t(x) - \nu_{n_T}(x)] g \circ \phi_t(x) dt \\
&= - \int_0^T dt \frac{d}{dt} (\chi \circ \tau_{n_T}(x, t) g \circ \phi_t(x)) + \nu_{n_T}(x) \int_0^T dt \chi' \circ \tau_{n_T}(x, t) g \circ \phi_t(x) \\
&= g(x) + \nu_{n_T}(x) \int_0^T dt \chi' \circ \tau_{n_T}(x, t) g \circ \phi_t(x),
\end{aligned}$$

where we have used (2.8) and the notation of the previous section. On the other hand

$$\int_0^T dt \chi' \circ \tau_{n_T}(x, t) g \circ \phi_t(x) = \mathbb{H}_{F^n(x), \chi'}(\mathcal{L}_{\mathbb{F}}^n g),$$

which, by the same argument as before, is uniformly bounded. Integrating (4.3) along the flow, yields

$$\overline{H}_T(g)(\phi_t(x)) - \overline{H}_T(g)(x) - \int_0^t g \circ \phi_s(s) ds = \mathcal{O}(\lambda^{-n_T}).$$

The Lemma follows remembering (3.2). \square

As discussed in the introduction Lemma 1.4 implies that g is a measurable coboundary. To study its regularity we must investigate the regularity of $\overline{H}_t(g)$. We will discuss only the first derivative, see Remark 1.5 for a discussion of this choice. Unfortunately, before getting to the real proof, we need to collect some technical facts.

4.3. Some technical preliminary facts.

First of all recall that given a one form $\omega(x) = \sum_{i=1}^2 a_i(x) dx_i$ and a diffeomorphism $G \in \mathcal{C}^1(\mathbb{T}^2, \mathbb{T}^2)$ the pullback of the form is given by

$$G^* \omega(x) = \sum_i a_i(G(x)) (D_x G)_{ij} dx_j,$$

while for a vector field v the pushforward is given by

$$G_* v(x) = D_{G^{-1}(x)} G \cdot v(G^{-1}(x)).$$

Using such a notation, by a direct computation, we have that, for each sector field $\mathbf{v} \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R}^2)$,

$$(4.4) \quad \begin{aligned} \mathbf{v}_i(x) \partial_{x_i} H_{x,t}(g) &= \int_0^t ds (\partial_{x_k} g) \circ \phi_s(x) (D_x \phi_s)_{k,i} \mathbf{v}_i(x) \\ &= \int_0^t ds (dg)_{\phi_s(x)} ((\phi_s)_* \mathbf{v}) \end{aligned}$$

where we have used the usual convention on the summation of repeated indexes. As done before, we would like to use the renormalizing dynamics F to transform the above expression in integrals over uniformly bounded domains. Inserting (4.2) in (4.4), changing variables and using repeatedly (1.5), (1.7) we obtain, for each $\mathbf{v} \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{R}^2)$,

$$(4.5) \quad \langle \mathbf{v}(x), \nabla H_{x,t}(g) \rangle = \int_0^{\tau_n(x,t)} ds \frac{[(F^{-n})^* dg]_{\phi_s \circ F^n(x)} ((\phi_s)_* F_*^n A_{x,s,n} \mathbf{v}(x))}{\nu_n(F^{-n} \circ \phi_s(F^n(x)))}.$$

The equation is now assuming a form similar to (2.4) and it seems then possible to express it again in terms of an appropriate transfer operator, this time acting on one forms. Yet, in order to do so effectively we need more informations on the relevant objects.

Next, we need to spell out the cocycle properties of τ_n and of the matrix

$$(4.6) \quad \Theta_{x,n}(s) := D_x F^n \cdot A_{x,s,n}.$$

Lemma 4.2. *For each $x \in \mathbb{T}^2$, $n, m \in \mathbb{N}$ and $s \in \mathbb{R}_>$ we have that $\Theta_{x,m}(s)$ is invertible and*

$$\begin{aligned} \tau_m(F^n(x), \tau_n(x, s)) &= \tau_{n+m}(x, s) \\ \Theta_{F^n(x), m}(\tau_m(F^n(x), s)) \cdot \Theta_{x,n}(s) &= \Theta_{x, n+m}(\tau_m(F^n(x), s)). \end{aligned}$$

Proof. Since DF and $D\phi_t$ are invertible by definition, the first assertion is equivalent to the fact that $A_{x, \tau_n(x,t), n}$ is invertible. If not, then there would exist $v \in \mathbb{R}^2$ such that

$$[\mathbb{1} + \nu_n(x)^{-1} V(x) \otimes \nabla \tau_n(x, t)] v = 0.$$

The above implies that $v = cV(x)$ for some $c \in \mathbb{R}$. Hence

$$-\nu_n(x) = \langle \nabla \tau_n(x, t), V(x) \rangle = \int_0^t \frac{d}{ds} \nu_n(\phi_s(x)) ds = \nu_n(\phi_t(x)) - \nu_n(x),$$

which is impossible since $\nu_n \neq 0$. The proof of the other equalities consists of a boring computation: by definition

$$\begin{aligned} \tau_m(F^n(x), \tau_n(x, t)) &= \int_0^{\tau_n(x,t)} \nu_m(\phi_s(F^n(x))) ds = \int_0^t \nu_m(\phi_{\tau_n(x,s)}(F^n(x))) \nu_n(\phi_s(x)) ds \\ &= \int_0^t \nu_m(F^n \circ \phi_s(x)) \nu_n(\phi_s(x)) ds = \int_0^t \nu_{n+m}(\phi_s(x)) ds = \tau_{n+m}(x, t), \end{aligned}$$

where we have used (1.7). Next, using again the definition,

$$\begin{aligned} \Theta_{F^n(x),m}(\tau_m(F^n(x),s)) \cdot \Theta_{x,n}(s) &= D_{F^n(x)} F^m \left[\mathbb{1} + \frac{1}{\nu_m(F^n(x))} V(F^n(x)) \otimes \nabla \tau_m(F^n(x),s) \right] \\ &\quad \times D_x F^n \left[\mathbb{1} + \frac{1}{\nu_n(x)} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x,s)) \right] \\ &= D_x F^{n+m} \left[\mathbb{1} + \frac{1}{\nu_{n+m}(x)} V(x) \otimes (D_x F^n)^* \nabla \tau_m(F^n(x),s) \right] \\ &\quad \times \left[\mathbb{1} + \frac{1}{\nu_n(x)} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x,s)) \right]. \end{aligned}$$

To continue we remark that, differentiating the first equality of the Lemma, yields

$$\nabla \tau_{n+m}(x,s) = (D_x F^n)^* \nabla \tau_m(F^n(x), \tau_n(x,s)) + \nu_m(\phi_{\tau_n(x,s)}(F^n(x))) \nabla \tau_n(x,s).$$

Thus,

$$\begin{aligned} \Theta_{F^n(x),m}(\tau_m(F^n(x),s)) \Theta_{x,n}(s) &= D_x F^{n+m} \left[\mathbb{1} + \frac{1}{\nu_{n+m}(x)} V(x) \otimes \nabla \tau_{n+m}(x, \tau_n^{-1}(x,s)) \right] \\ &\quad - \frac{\nu_m(\phi_s \circ F^n(x))}{\nu_{n+m}(x)} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x,s)) + \frac{1}{\nu_n(x)} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x,s)) \\ &\quad + \frac{\langle \nabla \tau_{n+m}(x, \tau_n^{-1}(x,s)), V(x) \rangle}{\nu_{n+m}(x) \nu_n(x)} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x,s)) \\ &\quad - \frac{\nu_m(\phi_s \circ F^n(x)) \langle \nabla \tau_n(x, \tau_n^{-1}(x,s)), V(x) \rangle}{\nu_{n+m}(x) \nu_n(x)} V(x) \otimes \nabla \tau_n(x, \tau_n^{-1}(x,s)) \Big]. \end{aligned}$$

To conclude note that, for each $k \in \mathbb{N}$ and $r \in \mathbb{R}_>$,

$$\langle \nabla \tau_k(x, r), V(x) \rangle = \int_0^r \frac{d}{ds} \nu_k(\phi_s(x)) ds = \nu_k(\phi_r(x)) - \nu_k(x).$$

Using such an equality in the above formula, and remembering that

$$\tau_{n+m}(x, \tau_n^{-1}(x,s)) = \tau_m(F^n(x), s),$$

the Lemma follows. \square

Finally we need bounds on $\Theta_{x,n}$.

Lemma 4.3. *There exists $C, \beta_1 > 0, \Lambda_0 > 1$ such that, for each $n \in \mathbb{N}$ and $x \in \mathbb{T}^2$, we have*

$$\|\Theta_{x,n}\|_{C^{r-2}((0,t), GL(2, \mathbb{R}))} \leq C \Lambda_0^n \tau_n^{-1}(x, t)^{\beta_1}.$$

Proof. By Lemma 4.1 and equation (4.2) it follows

$$\|\Theta_{x,n}\|_{C^{r-2}((0,t), GL(2, \mathbb{R}))} \leq C_{\#} \|D F^n\|_{C^r} \tau_n^{-1}(x, t)^{\beta} t^{\beta},$$

which implies the Lemma. \square

The above shows that (4.5) can be bounded in terms of norms of transfer operators acting on forms. Of course, we still have the same regularity problem that appeared in the previous sections, which will be dealt with in the same way, i.e. extending the dynamics to the Grassmanian. We can then define the operator, acting on one forms \mathfrak{g} defined on Ω by

$$(4.7) \quad \left[\widehat{\mathcal{L}}_{\mathbb{F}} \mathfrak{g} \right]_{(x,v)} = \frac{\|D_x F^{-1} v\| \|V(x)\|}{\|V \circ F^{-1}(x)\|} [(\mathbb{F}^{-1})^* \mathfrak{g}]_{(x,v)}.$$

Next we need to define again a more general functional. For each one form \mathbf{g} on Ω and vector field $w \in L^\infty(\mathbb{R}, \mathbb{R}^2)$, with compact support in $\mathbb{R}_>$, we define

$$\mathbb{H}_{x,w}^1(\mathbf{g}) = \int_{\mathbb{R}} \mathbf{g}_{\phi_s(x, \hat{V}(x))}((D_x \phi_s w(s), 0)) ds.$$

We can then write²⁴

$$(4.8) \quad \begin{aligned} \langle v, \nabla H_{x,t}(g) \rangle &= \int_0^{\tau_n(x,t)} ds \left[\hat{\mathcal{L}}_{\mathbb{F}}^n \pi^* dg \right]_{\mathbb{F}^n(x, \hat{V}(x))} ((D_{F^n(x)} \phi_s \Theta_{x,n}(s)v, 0)) \\ &= \mathbb{H}_{F^n(x), \Theta_{x,n}v}^1(\hat{\mathcal{L}}_{\mathbb{F}}^n \pi^* dg). \end{aligned}$$

To conclude we need the analogous of equation (3.7) and Lemma 3.1. Let \mathcal{A} be the set of one forms on Ω such that, for all $v \in \mathbb{R}^2$, $\mathbf{g}((0, v)) = 0$.

Lemma 4.4. *For each $t \in \mathbb{R}_>$, one form $\mathbf{g} \in \mathcal{A}$ and vector field $w \in L^\infty(\mathbb{R}, \mathbb{R}^2)$, with compact support, we have*

$$\mathbb{H}_{x, [\Theta_{x,n}^{-1} w] \circ \tau_n(x, \cdot)}^1(\mathbf{g}) = \mathbb{H}_{F^n(x), w}^1(\hat{\mathcal{L}}_{\mathbb{F}}^n \mathbf{g}).$$

Proof. By hypothesis $\mathbf{g}(v_1, v_2) = g_1(v_1)$ for some one form g_1 on \mathbb{T}^2 . Then, by a computation similar to (4.5), we have

$$\begin{aligned} \mathbb{H}_{x, [\Theta_{x,n}^{-1} w] \circ \tau_n(x, \cdot)}^1(\mathbf{g}) &= \int_{\mathbb{R}} [\phi_s^*(g_1)]_x([\Theta_{x,n}^{-1} w] \circ \tau_n(x, s)) ds \\ &= \int_{\mathbb{R}} \nu_n(F^{-n} \circ \phi_s(F^n(x)))^{-1}[(F^{-n})^* g_1]_{\phi_s(F^n(x))}((\phi_s)_* w(s), 0) ds. \end{aligned}$$

Next, note that if $\mathbf{g} \in \mathcal{A}$, then also $(\mathbb{F}^n)^* \mathbf{g} \in \mathcal{A}$, hence

$$\begin{aligned} \mathbb{H}_{x, [\Theta_{x,n}^{-1} w] \circ \tau_n(x, \cdot)}^1(\mathbf{g}) &= \int_{\mathbb{R}} \frac{[(\mathbb{F}^n)^* \mathbf{g}]_{\phi_s(F^n(x), \hat{V}(F^n(x)))}((\phi_s)_* w(s), 0)}{\nu_n(F^{-n} \circ \phi_s(F^n(x)))} ds \\ &= \mathbb{H}_{F^n(x), w}^1(\hat{\mathcal{L}}_{\mathbb{F}}^n \mathbf{g}). \end{aligned}$$

□

We have finally obtained all the facts needed to conclude the argument in a way similar to what we did in Section 3. Let us define appropriate classes of vector fields: $\widehat{\mathcal{D}}_{r,C}^s = \{w : C^r(\mathbb{R}_>, \mathbb{R}^2) : \|w\|_{C^r} \leq C\}$ and $\widehat{\mathcal{D}}_{r,C} = \{w : C_0^r(\mathbb{R}, \mathbb{R}^2) : \|w\|_{C^r} \leq C\}$. We are now ready to state the equivalent, in the present context, of the decomposition discussed in Lemma 3.1.

Lemma 4.5. *There exist $C_* > 0$ and $\Lambda_1 > 1$ such that, for each $n \in \mathbb{N}$, $t \in \mathbb{R}_>$, $v \in \mathbb{R}^2$, $\|v\| = 1$, and $g \in C^r(\mathbb{T}^2, \mathbb{R})$, there exist $K \in \mathbb{N}$, $\{n_i\}_{i=1}^K \subset \mathbb{N}$, $n_K = 0$, and $w_i \in \mathcal{D}_{r, C_* \Lambda_1^{n_i}}$, $w \in \mathcal{D}_{r, C_*}^s$ such that*

$$\langle v, \nabla H_{x,t}(g) \rangle = \sum_{i=1}^K \mathbb{H}_{F^{n_i}(x), w_i}^1(\hat{\mathcal{L}}_{\mathbb{F}}^{n_i} \mathbf{g}) + \mathbb{H}_{x,w}^1(\mathbf{g}).$$

Moreover, $\max\{|\text{supp } w|, |\text{supp } w_i|\} \leq 1$.

²⁴ Note that $(\mathbb{F}^{-n})^* \pi^* dg = \pi^*(F^{-n})^* dg$.

Proof. We use the exact same strategy of Lemma 3.1, only now we do not have to worry about the right extreme of the intervals since our test functions are automatically well behaved over there. We will discuss explicitly only the details which needs to be changed, since the proof given in Lemma 3.1 can be followed verbatim.

First of all note that $\mathbf{g} = \pi^* g \in \mathcal{A}$. Then, as in Lemma 3.1, but using Lemma 4.4, and using a ψ such that $\text{supp } \psi \subset (\delta, 2)$, $\psi|_{[2\delta, 1]} = 1$, $\psi^- = [1 - \psi]\mathbb{1}_{[0, 1]}$, we have

$$\langle v, \nabla \overline{H}_{x,t}(g) \rangle = \mathbb{H}_{F^{n_1}(x), \chi\psi^- \cdot \Theta_{x,n_1} v}^1(\widehat{\mathcal{L}}_{\mathbb{F}}^{n_1} \mathbf{g}) + \mathbb{H}_{F^{n_1}(x), \chi\psi \cdot \Theta_{x,n_1} v}^1(\widehat{\mathcal{L}}_{\mathbb{F}}^{n_1} \mathbf{g}).$$

Again, $\chi\psi\Theta_{x,n}v \in \mathcal{C}_0^{r-2}([0, 1], \mathbb{R})$, hence it is a good test function. Not so for $\chi\psi^- \Theta_{x,n_1}v$ owing to the discontinuity at zero. To treat the latter term notice that, by Lemmata 4.2 and 4.4, setting $n_1 - n_2 = m$, we have

$$\begin{aligned} \mathbb{H}_{F^{n_1}(x), \chi\psi^- \cdot \Theta_{x,n_1} v}^1(\widehat{\mathcal{L}}_{\mathbb{F}}^{n_1} \mathbf{g}) &= \mathbb{H}_{F^{n_2}(x), [\chi\psi^- \cdot \Theta_{F^{n_2}(x), m}^{-1} \Theta_{x,n_1} v] \circ \tau_m(F^{n_2}(x), \cdot)}^1(\widehat{\mathcal{L}}_{\mathbb{F}}^{n_2} \mathbf{g}) \\ &= \mathbb{H}_{F^{n_2}(x), [\chi\psi^-] \circ \tau_m(F^{n_2}(x), \cdot) \Theta_{x,n_2} v}^1(\widehat{\mathcal{L}}_{\mathbb{F}}^{n_2} \mathbf{g}). \end{aligned}$$

The above formula shows that the construction can be carried out exactly as before. The needed estimates on the functions w_i , w follows from Lemma 4.3, computations similar to the ones in Sub-Lemma 3.2 and noticing that $\|\tau_m^{-1}\|_{\mathcal{C}^r} \leq C_{\#} \Lambda_2^m$ for some $\Lambda_2 > 1$. \square

4.4. Proof of the second main result.

Having shown that the problem can be cast in a setting completely analogous to the one already discussed, we are now ready to conclude.

Proof of Theorem 2.12. This is the same argument carried out in the proof of Theorem 2.8, only now we need the spectral picture for the operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ on an different space $\widehat{\mathcal{B}}^{p,q}$, $p + q \leq r - 2$. In appendix D we show that there exists a Banach space $\widehat{\mathcal{B}}^{p,q}$ on which $\widehat{\mathcal{L}}_{\mathbb{F}}$ has spectral radius $\rho > 0$ and essential spectral radius bounded by $\lambda^{-\min\{p,q\}}\rho$. Also the functionals $\mathbb{H}_{x,w}^1$, for $w \in \mathcal{C}_0^r$, are bounded by

$$|\mathbb{H}_{x,w}^1(\mathbf{g})| \leq C_{\#} \|w\|_{\mathcal{C}^{r-2}} \|\mathbf{g}\|_{p,q}.$$

Thus, using the spectral decomposition of $\widehat{\mathcal{L}}_{\mathbb{F}}$ it follows that, if r is large enough, we can choose p, q so that $\lambda^{-\min\{p,q\}}\rho =: e^{\beta_{\text{ess}}} < \Lambda_1^{-1}$. It follows that if dg belongs to the kernel of all the distributions corresponding to the point spectrum of $\widehat{\mathcal{L}}_{\mathbb{F}}$, then Lemma 4.5 implies that

$$|\nabla H_{x,t}(g)| \leq C_{\#} \sum_{i=1}^K e^{\beta_{\text{ess}} n_i} \Lambda_1^{n_i} \|g\|_{\mathcal{C}^r} + C_{\#} \|g\|_{\mathcal{C}^1} \leq C_{\#} \|g\|_{\mathcal{C}^r}.$$

This implies that the $\overline{H}_{x,t}(g)$ are equicontinuous functions of x , hence there exists $\{t_j\}$ such that $\overline{H}_{x,t_j}(g)$ converges uniformly to a Lipschitz function. We have thus showed that $\overline{H}_{x,t}(g)$ has a convergent subsequence to a Lipschitz function h which satisfies (1.11), hence g is a Lipschitz coboundary. \square

APPENDIX A. A LITTLE CLASSIFICATION

Here we provide proof of the partial classification of the flows that satisfy our conditions.

Proof of Lemma 1.1. The map F is topologically conjugated to a linear automorphism [32, Theorem 18.6.1]. Such conjugation shows that the flow is topologically orbit equivalent to a rigid rotation. Hence one can choose a global Poincaré section and the associated Poincaré map. Such a map will have a rotation number determined by the foliation of the total automorphism, which a straightforward computation shows to have the claimed property.

On the contrary, if ϕ_t has no fix points nor periodic orbits, then there exists a global section uniformly transversal to the flow (see [42] for the original work, or [25] for a brief history of the problem and references) and the associated Poincaré map is a $\mathcal{C}^{1+\alpha}$ map of the circle with irrational rotation number ω . To claim that the Poincaré map is conjugated to a rigid rotation requires however some regularity. In particular, if $\alpha \geq 1$, then Denjoy Theorem [32, Theorem 12.1.1] implies that the Poincaré map is topologically conjugated to a rigid rotation. If ω is Diophantine, then for $\alpha \geq 2$ it is possible to show that the conjugation is \mathcal{C}^β for all $\beta < \alpha$, [30]. Then, if ω satisfies property (1.2), we can view the linear flow as the stable leaves of a total automorphism. We then obtain a \mathcal{C}^β Anosov map with the wanted properties by conjugation. \square

APPENDIX B. EXAMPLES

Lemma 1.1 shows that the flows to which our theory applies must necessarily enjoy several properties, the reader might be left wondering if such flows exist at all (a part, of course, for the trivial one consisting in rigid rotations).

To construct examples the simplest thing is to reverse the logic and start with a \mathcal{C}^r Anosov map which is orientation preserving. Given such a map, we have an associated stable distribution. If we choose any strictly positive function $\mathcal{N} \in \mathcal{C}^r(\mathbb{T}^2, \mathbb{R})$ there are only two fields V such that $V(x) \in E^s(x)$ for all $x \in \mathbb{T}^2$ and $\|V(x)\| = \mathcal{N}(x)$, they correspond to the two possible orientations. We can then choose any of the two and we have, at the same time, an example that satisfies all our assumptions and a justification of such assumptions. Indeed, in general the distribution E^s of a \mathcal{C}^r Anosov map will be only $\mathcal{C}^{1+\alpha}$ with $\alpha \in (0, 1)$, [32]. Notice however that it is possible to have situations in which $\alpha > 1$ and yet F is not \mathcal{C}^1 conjugated to a toral automorphism [32, Exercise 19.1.5]. Of course, in the latter case the unstable foliation will be irregular [24, Corollary 3.3].

The above partially clarifies the applicability of our work. Nevertheless, other reasons of unhappiness persist. In particular all our discussion, up to this point, has been a bit abstract as we do not really understand how our theory works and which type of concrete objects it yields. To understand it better let us work out the linear case that, surprisingly, it is not completely trivial.

B.1. A “trivial” example.

For the reader convenience we discuss here the case in which F is linear and ϕ_t is generated by a constant vector field. As already mentioned in the introduction this is the analogous of the case, for the geodesic-horocycle flow setting, of compact manifolds of constant curvature. Hence it can be dealt with directly by representation theory (i.e. Fourier transform, in the present setting), without using the strategy put forward in this paper.

Let $A \in SL(2, \mathbb{N})$; let $F_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the Anosov map defined by $F_A(x) \doteq Ax \bmod 1$. Since $\det(A) = 1$ the map is invertible, and has eigenvalues $\lambda_A, \lambda_A^{-1} \in \mathbb{R}$, with $\lambda_A > 1$. Let $V_A = (1, \omega)$ be the eigenvector associated to the eigenvalue λ_A^{-1} .

Note that ω is a quadratic irrational, as in Lemma 1.1. Let $\phi_t(x) = x + tV_A \pmod{1}$. In this case by applying Fourier transform to equation (1.11) we obtain, for $k \in \mathbb{Z}^2$, calling \hat{f}_k the Fourier coefficients of f ,

$$i2\pi\langle V, k \rangle \sum_{k \in \mathbb{Z}^2} \hat{h}_k e^{2\pi i\langle k, x \rangle} = \sum_{k \in \mathbb{Z}^2} \hat{g}_k e^{2\pi i\langle k, x \rangle}.$$

Note that we have the trivial obstruction $\hat{g}_k = 0$. If this is satisfied, note that $\langle V, k \rangle = k_1 + \omega k_2 \neq 0$ for all $(k_1, k_2) \in \mathbb{Z}^2 \setminus \{0\}$, since ω is irrational. Thus, we can write

$$\hat{h}(k) = -i \frac{\hat{g}(k)}{2\pi\langle V, k \rangle}.$$

Since ω is a quadratic irrational, it is well known (e.g. by using standard results on continuous fractions) that $|\langle V, k \rangle| \geq C_{\#} \|k\|^{-1}$. Hence, if $g \in W^{r,2}$ (the Sobolev space with the first r derivatives in L^2), then $h \in W^{r-1,2}$. In particular, if $g \in \mathcal{C}^\infty$, then $h \in \mathcal{C}^\infty$.

That is, in this example all the aforementioned distributions do not exist and the only obstruction is the trivial one: the one given by the invariant measure. If such an obstruction is satisfied (i.e. $\text{Leb}(g) = 0$), then the ergodic averages are bounded and g is a coboundary with the maximal regularity one can expect.

Nevertheless, it is very interesting to apply to this example also our strategy. This will give us a feeling for what might happen in general. First, let us change coordinates in Ω . One convenient choice is $\theta(x, s) = (x, v(s))$ with $v(s) = (1, s)(1 + s^2)^{-\frac{1}{2}}$ and $s < 0$. In this co-ordinates we have that the set Ω , defined just before (2.7), reads $\mathbb{T}^2 \times [-\beta, -\alpha]$ for some $0 < \alpha < \beta$. Also, calling $\hat{\mathbb{F}} = \theta^{-1} \circ \mathbb{F} \circ \theta$, we have

$$\begin{aligned} \hat{\mathbb{F}}(x, s) &= (F(x), \psi(s)) \\ \psi(s) &= \frac{c + sd}{a + sb}; \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \end{aligned}$$

Also we have

$$\psi^{-1}(s) = \frac{as - c}{d - bs}.$$

The map ψ^{-1} is a contracting map with derivative $(\psi^{-1})'(s) = (d - bs)^{-2}$ and unique fixed point \bar{s} in $[-\beta, -\alpha]$. The smallest eigenvalue of A is given by $\bar{\nu} = (d - \bar{s}b)^{-1}$ and the corresponding eigenvector is $\bar{v} = (1, \bar{s})$. Setting, as usual, $\theta^* \mathbf{g} = \mathbf{g} \circ \theta$, for $\mathbf{g} \in \mathcal{C}^0(\Omega, \mathbb{C})$, and the multiplication operator $\Xi \mathbf{g}(x, s) = \frac{\sqrt{1+s^2}}{\|V(x)\|} \mathbf{g}(x, s)$ let us define

$$\mathcal{L}_{\hat{\mathbb{F}}} = (\theta^*)^{-1} \Xi^{-1} \mathcal{L}_{\hat{\mathbb{F}}} \Xi \theta^*.$$

By direct computation it follows²⁵

$$\mathcal{L}_{\hat{\mathbb{F}}} \mathbf{g}(x, s) = \mathbf{g} \circ \hat{\mathbb{F}}^{-1}(x, s)(d - bs).$$

As the two operators are conjugated, it suffices to study the spectrum of $\mathcal{L}_{\hat{\mathbb{F}}}$. If we look for eigenvalues of the form $\mathbf{g}(x, s) = g(x)f(s)$ we have that $\mathcal{L}_{\hat{\mathbb{F}}} \mathbf{g} = \mu \mathbf{g}$ reads

$$\mu g(x)f(s) = g \circ F^{-1}(x)f(\psi^{-1}(s))(d - bs).$$

By the above computation in Fourier modes, it follows that $g(x) = 1$, hence we need

$$\mu f(s) = f(\psi^{-1}(s))(d - bs).$$

²⁵ Remember that $s < 0$, hence $d - sb > 0$.

Iterating the above relation yields

$$f(s) = \mu^{-n} \prod_{k=0}^{n-1} (d - b\psi^{-k}(s)) f(\psi^{-n}(s)).$$

Since $\psi^{-n}(s)$ converges to \bar{s} there are two possibilities; either $f(\bar{s}) \neq 0$ or $f(\bar{s}) = 0$. In the first case we can assume, without loss of generality, that $f(\bar{s}) = 1$. Then

$$f(s) = \prod_{k=0}^{\infty} \mu^{-1} (d - b\psi^{-k}(s)) = \prod_{k=0}^{\infty} \mu^{-1} (d - b\bar{s} + b[\psi^{-k}(\bar{s}) - \psi^{-k}(s)]),$$

provided the product converges. If we choose $\mu = d - b\bar{s} = \bar{\nu}^{-1}$, then, for any $\tau > \bar{\nu}^2$ we have

$$\prod_{k=0}^{\infty} \mu^{-1} (\mu + b[\psi^{-k}(\bar{s}) - \psi^{-k}(s)]) = e^{\sum_{k=0}^{\infty} \mathcal{O}(\tau^k)}$$

which shows the convergence.

Next, consider the case $f(\bar{s}) = 0$. In this case it matters the order of the zero. It is then natural to look for solutions of the form $f(s) = (s - \bar{s})^p M(s)$, $M(\bar{s}) = 1$, for $p \in \mathbb{N}$, then

$$\begin{aligned} \mu(s - \bar{s})^p M(s) &= (d - bs)(\psi^{-1}(s) - \psi^{-1}(\bar{s}))^p M \circ \psi^{-1}(s) \\ &= (d - bs)^{-p+1} (d - b\bar{s})^{-p} (s - \bar{s})^p M \circ \psi^{-1}(s). \end{aligned}$$

Iterating again we have

$$M(s) = \prod_{k=0}^{\infty} \mu^{-1} (\bar{\nu}^{-1} + b(\psi^{-k}(\bar{s}) - \psi^{-k}(s)))^{-p+1} \bar{\nu}^p.$$

The above is convergent provided $\mu = \bar{\nu}^{2p-1}$. Hence $\sigma(\mathcal{L}_{\widehat{\mathbb{F}}}) \subset \{\bar{\nu}^{2k-1}\}_{k \in \mathbb{N}}$.

To check that we found all the eigenvalues, we can use the trace formula²⁶

$$(B.1) \quad \text{Tr } \mathcal{L}_{\widehat{\mathbb{F}}}^n = \sum_{(x,s) \in \text{Fix } \widehat{\mathbb{F}}^n} \frac{|\det(D\widehat{\mathbb{F}}^n)| \prod_{j=0}^{n-1} (d - b\psi^{-j}(s))}{|\det(\mathbf{1} - D\widehat{\mathbb{F}}^n)|}.$$

To simplify then computation of the periodic orbits, let us choose a specific example:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then the only fixed point of $\widehat{\mathbb{F}}$ is $(0, \bar{s})$, and

$$\text{Tr } \mathcal{L}_{\widehat{\mathbb{F}}} = \frac{|\det(D\widehat{\mathbb{F}})|(1 - \bar{s})}{|\det(\mathbf{1} - D\widehat{\mathbb{F}})|}.$$

Since

$$D\widehat{\mathbb{F}}(0, \bar{s}) = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & (1 - \bar{s})^2 \end{pmatrix}$$

²⁶ This is what one expects to hold by a formal computation. We are not aware of a proof that applies to the present case, but it should be possible to obtain it by moderate modifications of existing arguments.

we have

$$\mathrm{Tr} \mathcal{L}_{\widehat{\mathbb{F}}} = \frac{(1 - \bar{s})^3}{(1 - \bar{s})^2 - 1} = \frac{\bar{\nu}^{-1}}{1 - \bar{\nu}^2} = \sum_{k=0}^{\infty} \bar{\nu}^{2k-1}.$$

We can then be confident that we have identified all the spectrum of $\mathcal{L}_{\mathbb{F}}$. To conclude and be able to apply our theory we need to compute the eigenfunctions of $\mathcal{L}'_{\mathbb{F}}$. By definition

$$\mathcal{L}'_{\widehat{\mathbb{F}}} \mathbf{h}(x, s) = \mathbf{h} \circ \widehat{\mathbb{F}}(x, s)(1 - \psi(s))(1 - s)^2.$$

Again we look for solutions of the type $\mathbf{h}(x, s) = h(s)$. Thus we have to solve

$$h_k(\psi(s))(1 - \psi(s))(1 - s)^2 = \bar{\nu}^{2k-1} h_k(s)$$

where h_k are distributions. That is, for all $\varphi \in \mathcal{C}^\infty$ we want

$$\int h_k(s)(1 - s)\varphi \circ \psi^{-1}(s)ds = \bar{\nu}^{2k-1} \int h_k(s)\varphi(s)ds.$$

Note that if $h = \sum_{j=1}^k c_j \delta^{(j)}(s - \bar{s})$, then

$$\int h(s)(1 - s)\varphi \circ \psi^{-1}(s)ds = \sum_{j=1}^k c'_j \varphi^{(j)}(\bar{s}).$$

That is the vector spaces $\mathbb{V}_k = \mathrm{span}\{\delta(s - \bar{s}), \dots, \delta^{(k)}(s - \bar{s})\}$ are invariant. It follows that $h_0 \in \mathbb{V}_0$ and $h_k \in \mathbb{V}_k \setminus \mathbb{V}_{k-1}$ for $k > 0$.

Note that, as remarked just before Theorem 2.8, the projection π_* is not one-one. In the present case $O_k = \pi_* \mathbf{h}_k$ are all proportional to Lebesgue. Hence as we already saw, there are no non trivial obstructions. It is however interesting that this does not imply that the spectrum of $\mathcal{L}'_{\mathbb{F}}$ consists only of zero and $e^{h_{\mathrm{top}}}$.

B.2. Few considerations on the general case.

Given the previous discussion, a natural question is if there are or not cases in which non trivial obstructions exists. We conjecture that the answer is positive, but it is not so easy to give an explicit example. Here we content ourselves with few comments.

Consider the symplectic maps studied in [36]:

$$(B.2) \quad F_\alpha(x, y) = \left(2x + y - \frac{\alpha}{2\pi} \sin 2\pi x, x + y - \frac{\alpha}{2\pi} \sin 2\pi x \right).$$

Also choose $\mathcal{N} = 1$ and call V_α the vector field. Note that F_0 is the linear total automorphism discussed in the previous section. On the other hand, when $\alpha = 1$ the map is no longer Anosov. In fact we have a map of the class studied in [37] where it is shown that the decay of correlations with respect to Lebesgue, is only polynomial. In particular this shows that the Ruelle transfer operator cannot have a spectral gap. This is suggestive, although the relevance of such a fact in the present context is unclear since we are interested in a different operator.

Using the same co-ordinates than in the previous section we can reduce ourselves to the study of the map

$$\begin{aligned} \widehat{\mathbb{F}}_\alpha(x, s) &= (F_\alpha(x), \psi_\alpha(x, s)) \\ \psi_\alpha(x, s) &= \frac{1 - \alpha \cos 2\pi x + s}{2 - \alpha \cos 2\pi x + s}. \end{aligned}$$

and the transfer operator

$$\mathcal{L}_{\widehat{\mathbb{F}}_\alpha} \mathbf{g}(x, s) = \mathbf{g} \circ \widehat{\mathbb{F}}_\alpha^{-1}(x, s)(1 - \alpha \sin 2\pi x - s).$$

For small α the above operator is a perturbation of $\mathcal{L}_{\widehat{\mathbb{F}}_0}$ the spectrum of which we have computed. So by the perturbation theory in [33] it will have near by eigenvalues. Also, there is no obvious reason to expect that the relative eigenvectors will have all the same projection. Hence we expect that as soon as $\alpha \neq 0$ non trivial obstructions will appear. Unfortunately, the second eigenvalue will be very close to $\bar{\nu} < 1$, so it will not have any influence on the growth of ergodic averages. In the best case, it will appear an obstruction to the existence of a Lipschitz coboundary. To have ergodic averages with a growth t^α , $\alpha \notin \{0, 1\}$ seems therefore to be related to having a transfer operator, associated to the measure of maximal entropy, with eigenvalues close to one. We believe this to be possible, yet to our knowledge no such example is currently known.²⁷

APPENDIX C. ANISOTROPIC BANACH SPACES: DISTRIBUTIONS

In this section we first construct the Banach spaces used in Section 3, then we discuss the relation with the Banach spaces constructed in [28], finally, we prove Proposition 2.7 and show that \mathbb{H} is a bounded functional.

The construction of the Banach spaces are based on the definition of appropriate norms. The Banach spaces are then obtained by closing $\mathcal{C}^r(\Omega, \mathbb{C})$ with respect to such norms.²⁸ The basic idea is to control not the functions themselves but rather their integrals along curves close to the stable manifolds. Hence the first step is to define the set of relevant curves.²⁹ To do so we need to fix $\delta \in (0, 1/2)$ and $K \in \mathbb{R}_>$.

Definition C.1 (Admissible leaves). *Given $r \in \mathbb{R}_>$, an admissible leave $W \subset \mathbb{T}^2$ is a \mathcal{C}^r curve with length in the interval $[\delta/2, \delta]$. We require that there exists a parametrization $\omega : [0, 1] \rightarrow W$ of such a curve such that $\omega'(\tau) \in C^s(\omega(\tau))$, for all $\tau \in [0, 1]$, and $\|\omega\|_{\mathcal{C}^r([0, 1], \mathbb{T}^2)} \leq K$. Moreover we ask $(\omega(\tau), \omega'(\tau)\|\omega'(\tau)\|^{-1}) \in \Omega$, that is the curves have all the chosen orientation. We call Σ the set of admissible curves where to any $W \in \Sigma$ is associated a parametrisation ω_W satisfying the properties mentioned above.*

The above set is not empty and contains pieces of stable manifolds, if K is large enough, since the stable manifolds are uniformly \mathcal{C}^r , [32]. The basic fact about admissible curves is that if $W \in \Sigma$, then, for each $n \in \mathbb{N}$, $F^{-n}W \subset \cup_{i=1}^{N_n} W_i$ for some finite set $\{W_i\}_{i=1}^{N_n} \subset \Sigma$. This is quite intuitive but see [27] for a detailed proof in a more general setting.

Next, we define the integral of an element $\mathbf{g} \in \mathcal{C}^r(\Omega, \mathbb{C})$ along an element $W \in \Sigma$ against any $\varphi \in \mathcal{C}^0(W, \mathbb{C})$:

$$(C.1) \quad \int_W \varphi \mathbf{g} := \int_0^1 ds \varphi \circ \omega_W(s) \cdot \mathbf{g}(\omega_W(s), \omega'_W(s)\|\omega'_W(s)\|^{-1})\|\omega'_W(s)\|.$$

²⁷ Although some hope is given by the construction of generic examples, for the operator associated to the SRB measure, with spectrum different from $\{0, 1\}$ by Alexander Adam [1], following a suggestion by Frédéric Naud.

²⁸ We consider complex valued functions because we are interested in having nice spectral theory.

²⁹ In fact, in the simple case at hand, we could consider directly pieces of stable manifolds. We do not do it to make easier to use already existing results.

Also, given $W \in \Sigma$ and $\varphi : W \rightarrow \mathbb{R}$ we set, for all $s \leq r$,

$$(C.2) \quad \|\varphi\|_{C^s(W, \mathbb{R})} \doteq \|\varphi \circ \omega_W\|_{C^s([0,1], \mathbb{R})}.$$

We are now ready to define the relevant semi-norms:³⁰

$$(C.3) \quad \|g\|_{p,q} := \sup_{W \in \Sigma} \sup_{|\alpha| \leq p} \sup_{\varphi \in C_0^{q+|\alpha|}(W, \mathbb{C})} \int_W \varphi \cdot \partial^\alpha(g),$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the usual multi-index and 1, 2 refer to the x co-ordinate while 3 refers to v .³¹ It is easy to check that the $\|\cdot\|_{p,q}$ are indeed semi-norms on $C^r(\Omega, \mathbb{C})$.

Definition C.2 ($\mathcal{B}^{p,q}$ spaces). *Let $p \in \mathbb{N}^*$, $q \in \mathbb{R}$, $p + q \leq r$ and $q > 0$. We define $\mathcal{B}^{p,q}$ to be the closure of $C^r(\Omega, \mathbb{C})$ with respect to the semi-norm $\|\cdot\|_{p,q}$.*³²

Remark C.3. *Note that $\|g\|_{p,q} \leq \|g\|_{C^{p+q}}$.*

The Banach spaces defined above are well suited for the tasks at hand but, unfortunately, they are not exactly the one introduced in [28] where a more general theory is put forward. To avoid having to develop the theory from scratch, it is convenient to show how to relate the present setting to the one in [28]. To this end let us briefly recall the construction in [28], then we will explain the relation with the present one. This will allow us to apply the general results in [28] to the present context.

We start by recalling, particularizing them to our simple situation, the basic objects used in [28]: the r times differentiable sections \mathcal{S}^r of a line bundle over the Grassmannian of one dimensional subspaces. More precisely, let $\mathcal{G} = \{(x, E)\}$ where $x \in \mathbb{T}^2$ and $E \subset \mathbb{R}^2$ is a linear one dimensional subspace, then $h \in \mathcal{S}^r$ is a C^r map $(x, E) \rightarrow E^*$.³³ Note that there is a strict relation between \mathcal{S}^r and $C^r(\Omega, \mathbb{C})$: for each $(x, v) \in \Omega$ let $E_v = \{\mu v\}_{\mu \in \mathbb{R}}$, then for each $h \in \mathcal{S}^r$ define $i : \mathcal{S}^r \rightarrow C^r(\Omega, \mathbb{C})$ by

$$[ih](x, v) = h(x, E_v)(v).$$

The important fact is that the elements of \mathcal{S}^r , when restricted to the tangent bundle of W , are volume forms on W , hence can be integrated. Let us be explicit: given $W \in \Sigma$, $h \in \mathcal{S}^r$ and $\varphi \in C^0(\mathbb{T}^2, \mathbb{C})$, by (C.1) and [28, Section 2.2.1] we have

$$(C.4) \quad \int_W \varphi h := \int_0^1 ds \varphi \circ \omega_W(s) h(\omega_W(s), E_{\omega'_W(s)})(\omega'_W(s)) = \int_W \varphi ih.$$

Finally, note that the norm in [28] is also given by integrals along curves in Σ . Accordingly, if $h, \tilde{h} \in \mathcal{S}^r$ differ only for (x, E) such that E does not belong to $C^s(x)$, then any norm of the difference based on integrations along curves in Σ will be zero. Thus, with respect to any such norm i will be an isomorphism. The readers can then check that the norms defined in [28] are equivalent to $\|ih\|_{p,q}$. Thus our spaces $\mathcal{B}^{p,q}$ are isomorphic, as a Banach space, to the ones defined in [28]

³⁰ By $C_0^s(W, \mathbb{C})$ we mean the C^s functions with support contained in $\text{Int}(W)$. The fact that the test functions must be zero at the boundary of W is essential for the following arguments.

³¹ To be more explicit, if we choose a chart $v = (\cos \theta, \sin \theta)$, then α_3 refers to the derivative with respect to θ .

³² To be precise the elements of $\mathcal{B}^{p,q}$ are the equivalence classes determined by the equivalence relation $h \sim \tilde{h}$ if and only if $\|h - \tilde{h}\|_{p,q} = 0$.

³³ To be precise, since we are going to do spectral theory, we should consider the complex dual. We do not insist on this since the complexification is totally standard.

that, not by chance, are called there $\mathcal{B}^{p,q,1}$ (the superscript 1 refers there to the fact that, as we will see briefly, in the present language we do not need to have a weight in the transfer operator). Finally, we have to understand how the operator $\mathcal{L}_{\mathbb{F}}$ reads in the corresponding language of [28]. To this end it is useful to introduce the operator $\Xi : \mathcal{C}^r(\Omega, \mathbb{C}) \rightarrow \mathcal{C}^r(\Omega, \mathbb{C})$ defined by

$$(\Xi g)(x, v) := g(x, v) \|V(x)\|.$$

Note that, by the assumptions of Definition 2.3, Ξ is invertible and both the operator and its inverse can be extended to a continuous operator on $\mathcal{B}^{p,q}$. It then follows by equations (C.1), (C.4), (2.10), and [28, Section 3.2] that, for all $W \in \Sigma$ and $\varphi \in \mathcal{C}^0(\mathbb{T}^2, \mathbb{C})$, we have

$$\begin{aligned} \int_W \varphi \mathbf{i}^{-1} \Xi^{-1} \mathcal{L}_{\mathbb{F}} \Xi \mathbf{i} h &= \int_0^1 ds \varphi(\omega_W(s)) (\Xi^{-1} \mathcal{L}_{\mathbb{F}} \Xi \mathbf{i} h)(\omega_W(s), \widehat{\omega}'_W(s)) \|\omega'_W(s)\| \\ &= \int_0^1 ds \varphi(\omega_W(s)) (\mathbf{i} h) \left(F^{-1} \omega_W(s), \frac{D_{\omega_W(s)} F^{-1} \omega'_W(s)}{\|D_{\omega_W(s)} F^{-1} \omega'_W(s)\|} \right) \|D_{\omega_W(s)} F^{-1} \omega'_W(s)\| \\ &= \int_{F^{-1}W} \varphi \circ F \mathbf{i} h = \int_{F^{-1}W} \varphi \circ F h = \int_W \varphi F_* h, \end{aligned}$$

where we used the notation $\widehat{\omega}'_W(s) = \omega'_W(s) \|\omega'_W(s)\|^{-1}$. Hence we conclude that

$$(C.5) \quad \mathbf{i}^{-1} \Xi^{-1} \mathcal{L}_{\mathbb{F}} \Xi \mathbf{i} h = F_* h := (F^{-1})^* h,$$

that is $\mathcal{L}_{\mathbb{F}}$ is conjugated to the push-forward of F on \mathcal{S}^r .

Proof of Proposition 2.7. Since (C.5) states that our operator is conjugated to the push-forward F_* , all the spectral properties of F_* , acting on $\mathcal{B}^{p,q,1}$, and $\mathcal{L}_{\mathbb{F}}$, acting on $\mathcal{B}^{p,q}$, coincide. It thus suffices to note that [28, Proposition 4.4, Theorem 5.1, Theorem 6.4] state that, for $q \in \mathbb{R}_>$, $p \in \mathbb{N}_>$ and $p+q \leq r$, F_* can be extended continuously to $\mathcal{B}^{p,q,1}$, that the logarithm of the spectral radius of F_* is given by the topological entropy (which is the maxim of the metric entropy), that the maximal eigenvalue is simple and F_* has a spectral gap and the essential spectral radius is bounded by $e^{h_{\text{top}}} \lambda^{-\min\{p,q\}}$. \square

We have thus seen that the operator $\mathcal{L}_{\mathbb{F}}$ acts very nicely on the spaces $\mathcal{B}^{p,q}$. The next important fact is that the functionals we are interested in are well behaved on such spaces.

Lemma C.4. *There exists $C > 0$ such that, for each $x \in \mathbb{T}^2$, $q \in \mathbb{R}_>$, $p \in \mathbb{N}_>$, $p+q \leq r$, and $\varphi \in \mathcal{C}_0^r(\mathbb{R}_>, \mathbb{R})$, $g \in \mathcal{C}^r(\Omega, \mathbb{R})$ we have*

$$|\mathbb{H}_{x,\varphi}(g)| \leq C |\text{supp } \varphi| \|g\|_{p,q} \|\varphi\|_{\mathcal{C}^{p+q}}.$$

Proof. Let us start by considering the case $\text{supp } \varphi \subset [a, a+\delta]$, for some $a > 0$. Then, $\{\phi_t(x)\}_{t \in [a, a+\delta]}$ is the re-parametrization of a curve W in Σ . To see it just consider the parametrization $\omega_W(s) = \phi_{a+\delta s}(x)$. Moreover, setting $\tilde{\varphi}(\phi_s(x)) = \varphi(s)$,³⁴ by (C.1) and (3.1)

$$\begin{aligned} \int_W \tilde{\varphi} g &= \int_0^1 ds \varphi \circ \phi_{a+\delta s}(x) g(\phi_{a+\delta s}(x), \widehat{V}(\phi_{a+\delta s}(x))) \|\widehat{V}(\phi_{a+\delta s}(x))\| \delta \\ &= \int_{\mathbb{R}} ds \varphi(s) (\Xi g) \circ \phi_s(x, \widehat{V}(x)) = \mathbb{H}_{x,\varphi}(\Xi g). \end{aligned}$$

³⁴ Note that, since the stable manifolds are uniformly \mathcal{C}^r , [32], $\|\tilde{\varphi}\|_{\mathcal{C}^r(W, \mathbb{R})} \leq C_{\#} \|\varphi\|_{\mathcal{C}^r(\mathbb{R}_>, \mathbb{R})}$.

Since the first quality on the left is exactly one of the functionals used in (C.3) to define the norm ($p = 0$) and Ξ^{-1} is a bounded operator on each space $\mathcal{B}^{p,q}$, we have

$$\|\mathbb{H}_{x,\varphi}(\mathbf{g})\| \leq C_{\#} \|\Xi^{-1}\|_{0,p} \|\varphi\|_{C^q} \|\mathbf{g}\|_{0,q} \leq C_{\#} \|\varphi\|_{C^{p+q}} \|\mathbf{g}\|_{p,q}.$$

The Lemma follows then by using a partition of unity. \square

APPENDIX D. ANISOTROPIC BANACH SPACES: CURRENTS

In this appendix we briefly describe the Banach spaces of currents used in our second results and sketch the needed facts. We will be much faster than in Appendix C and will omit several details as the construction is very similar and no essentially new ideas are present.

We consider the same set of admissible leaves detailed in Definition C.1. For each $W \in \Sigma$, let \mathcal{V}^q be the set of C^q vector fields compactly supported on W and with C^q norm bounded by one. Then, for each smooth one form \mathbf{g} defined on ω we define

$$(D.1) \quad \|\mathbf{g}\|_{p,q} := \sup_{W \in \Sigma} \sup_{|\alpha| \leq p} \sup_{\varphi \in \mathcal{V}^{q+|\alpha|}} \int_W [\partial^\alpha(\mathbf{g})](\varphi),$$

where the integral is defined as in the previous section.

Note that there exists a standard isomorphism \mathbf{i} from vector fields to one forms, so that $\mathbf{g}(\varphi) = \langle \mathbf{g}, \mathbf{i}(\varphi) \rangle$.³⁵ Thus the above norm is equivalent to the norm $\|\cdot\|_{p,q,1}$ used in [26]. Hence if we define $\widehat{\mathcal{B}}^{p,q}$ as the closure of the smooth one form with respect to the above norm, we obtain a space isomorphic to $\mathcal{B}^{p,q,1}$ of [26].

Unfortunately, the transfer operator used here differs from the one studied in [26] insofar it has a potential, which was absent in [26]. In principle, we should therefore prove the Lasota-Yorke inequality for our operator and compute the spectral radius for the present operator via a variational principle (as in [28]). Since such a computation is completely standard but a bit lengthy, we just state a partial result that suffices for our goals (in particular we do not bother computing exactly the spectral radius). Such a result follows by copying the computations made in [26] to obtain the Lasota-Yorke inequality. Such computations are exactly the same, apart from the need to keep track of the potential, which can be done easily:

- The operator $\widehat{\mathcal{L}}_{\mathbb{F}}$ extends continuously on $\widehat{\mathcal{B}}^{p,q}$, has spectral radius ρ and essential spectral radius strictly bounded by $\lambda^{-\min\{p,q\}} \rho$.
- For all $w \in C^r$ and $x \in \mathbb{T}^2$ and $p + q \leq r$ we have³⁶

$$|\mathbb{H}_{x,w}^1(\mathbf{g})| \leq C_{\#} |\text{supp } w| \|\mathbf{g}\|_{p,q} \|w\|_{C^{p+q}}.$$

The above two facts are all we presently needed.

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³⁵ See [26] for the relevant definition of scalar product between forms in the present context.

³⁶ This follows immediately from the definition of the norm.

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